A FIXED POINT METHOD TO THE GENERALIZED STABILITY OF A MIXED ADDITIVE AND QUADRATIC FUNCTIONAL EQUATION IN BANACH MODULES

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Abstract. In this paper we establish the general solution of the functional equation

$$\sum_{i=1}^{k} f\big(2x_i + \sum_{j=1, j\neq i}^{k} x_j\big) = (k+2) f\big(\sum_{i=1}^{k} x_i\big) + \sum_{i=1}^{k} f(-x_i) \quad (k \geq 2),$$

and use a fixed point method to prove its Hyers–Ulam–Rassias stability in Banach modules over a unital Banach algebra.

1. Introduction and preliminaries

In 1940, S.M. Ulam [?] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. The stability problem of functional equations originated from a question of Ulam [?] concerning the stability of group homomorphisms:

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In 1941, D. H. Hyers [?] considered the case of approximately additive functions $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive function satisfying

$$\|f(x) - L(x)\| \leq \epsilon.$$

2000 *Mathematics Subject Classification.* Primary: 39B72; Secondary 47H09.

Key words and phrases. Hyers–Ulam–Rassias stability, quadratic function, additive function, Banach module, unital Banach algebra, generalized metric space, fixed point.

T. Aoki [?] and Th.M. Rassias [?] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The above inequality has provided a lot of influence in the development of what is now known as a *Hyers–Ulam–Rassias stability* of functional equations. P. Găvruta [?] provided a further generalization of the Th.M. Rassias' theorem. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the Hyers–Ulam–Rassias stability to a number of functional equations and mappings (see [?], [?], [?], [?], [?], [?], [?], [?], [?] and [?]–[?]). We also refer the readers to the books [?], [?], [?], [?] and [?].

Quadratic functional equations were used to characterize inner product spaces [?, ?, ?]. A square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is related to a symmetric bi-additive function [?, ?]. The functional equation (??) is called a *quadratic functional equation.* In particular, every solution of the quadratic equation (??) is said to be a *quadratic function.* It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all x (see [?, ?]. The bi-additive function B is given by

$$B(x, y) = \frac{1}{4}\Big[f(x+y) - f(x-y)\Big].$$

A Hyers–Ulam–Rassias stability problem for the quadratic functional equation (**??**) was proved by Skof for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space (see [**?**]). Cholewa [**?**] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In the paper [**?**], Czerwik proved the Hyers–Ulam–Rassias stability of the quadratic functional equation (**??**). Grabiec [**?**] has generalized these results mentioned above. Jun and Lee [**?**] proved the Hyers–Ulam–Rassias stability of a Pexiderized quadratic equation.

Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a *generalized metric* on E if d satisfies

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in E$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.2. [**?**] *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a non-negative integer n_0 such that

1. *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
2. *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
3. *y^* is the unique fixed point of J in the set $Y = \{\, y \in E : d(J^{n_0} x, y) < \infty \,\}$;*
4. *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In this paper, we deal with the following functional equation deriving from quadratic and additive functions:

$$\sum_{i=1}^{k} f(2x_i + \sum_{j=1, j\neq i}^{k} x_j) = (k+2) f\big(\sum_{i=1}^{k} x_i\big) + \sum_{i=1}^{k} f(-x_i) \tag{1.4}$$

where $k \geq 2$ is a fixed integer.

It is easy to see that the function $f(x) = ax^2 + bx$ is a solution of the functional equation (**??**). In this paper we establish the general solution of (**??**), and using the fixed point method (see [**?**, **?**, **?**]), we prove the Hyers–Ulam–Rassias stability of (**??**). The first systematic study of fixed point theorems in nonlinear analysis is due to G. Isac and Th.M. Rassias; cf. [**?**].

2. Solutions of Eq. (??)

Throughout this section, X and Y will be real vector spaces. Before proceeding the proof of Theorem **??**, which is the main result in this section, we shall need the following two lemmas.

Lemma 2.1. *If an odd function $f : X \longrightarrow Y$ satisfies the functional equation* (**??**)*, then f is additive.*

Proof. Setting $x_1 = x$, $x_2 = y$ and $x_3 = ... = x_k = 0$ in (**??**) and using the oddness of f, we get

$$f(2x + y) + f(x + 2y) = 4f(x + y) - f(x) - f(y) \tag{2.1}$$

for all $x, y \in X$. Let $y = 0$ in (**??**), so

$$f(2x) = 2f(x) \tag{2.2}$$

for all $x \in X$. Replacing x and y by $x+y$ and $-y$ in (**??**), respectively, and using (**??**) and the oddness of f, we get

$$f(2x + y) + f(x - y) + f(x + y) = 4f(x) + f(y) \tag{2.3}$$

for all $x, y \in X$. Replacing x and y by y and x in (**??**), respectively, and using the oddness of f, we have

$$f(x + 2y) - f(x - y) + f(x + y) = f(x) + 4f(y) \tag{2.4}$$

for all $x, y \in X$. Adding (**??**) to (**??**) and using (**??**), we infer that

$$f(x + y) = f(x) + f(y) \tag{2.5}$$

for all $x, y \in X$. Therefore the function $f : X \to Y$ is additive. □

Lemma 2.2. *If an even function $f : X \to Y$ satisfies the functional equation* (**??**)*, then f is quadratic.*

Proof. Setting $x_1 = x$, $x_2 = y$ and $x_3 = ... = x_k = 0$ in (**??**) and using the evenness of f, we get

$$f(2x + y) + f(x + 2y) = 4f(x + y) + f(x) + f(y) \tag{2.6}$$

for all $x, y \in X$. Putting $x = y = 0$ in (**??**), we get $f(0) = 0$. Setting $y = 0$ in (**??**), we obtain that $f(2x) = 4f(x)$ for all $x \in X$. Replacing x and y by $x + y$ and $-y$ in (**??**), respectively, and using the evenness of f, we get

$$f(2x + y) + f(x - y) = f(x + y) + 4f(x) + f(y) \tag{2.7}$$

for all $x, y \in X$. Replacing x and y by y and x in (**??**), respectively, and using the evenness of f, we have

$$f(x + 2y) + f(x - y) = f(x + y) + 4f(y) + f(x) \tag{2.8}$$

for all $x, y \in X$. Adding (**??**) to (**??**) and using (**??**), we infer that

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{2.9}$$

for all $x, y \in X$. Therefore the function $f : X \to Y$ is quadratic. □

Theorem 2.3. *A function $f : X \to Y$ satisfies* (**??**) *for all $x_1, ..., x_k \in X$ if and only if there exist a symmetric bi-additive function $B : X \times X \to Y$ and an additive function $A : X \to Y$ such that $f(x) = B(x, x) + A(x)$ for all $x \in X$.*

Proof. If there exist a symmetric bi-additive function $B : X \times X \to Y$ and an additive function $A : X \to Y$ such that $f(x) = B(x,x) + A(x)$ for all $x \in X$, then by a simple computation one can show that the functions B and A satisfy the functional equation (**??**). So f satisfies (**??**).

Conversely, we decompose f into the even part and the odd part by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in X$. It is easy to show that the functions f_e and f_o satisfy (**??**). Hence by Lemmas **??** and **??** we achieve that the functions f_e and f_o are quadratic and additive, respectively. Therefore there exists a symmetric bi-additive function $B : X \times X \to Y$ such that $f_e(x) = B(x,x)$ for all $x \in X$ (see [**?**]). So

$$f(x) = B(x,x) + A(x)$$

for all $x \in X$, where $A(x) = f_o(x)$ for all $x \in X$. □

3. Hyers–Ulam–Rassias stability of Eq. (??)

Throughout this section, let B be a unital Banach algebra with norm $|.|$, $B_1 = \{\, a \in B : |a| = 1 \,\}$, $\mathbb{X}$ and $\mathbb{Y}$ be left Banach B-modules, and let $k \geq 2$ be a fixed integer. In this paper using an idea of Găvruta [**?**] we prove the stability of (**??**) in the spirit of Hyers, Ulam and Rassias. For convenience, we use the following abbreviation for a given function $f : \mathbb{X} \to \mathbb{Y}$:

$$D_a f(x_1, ..., x_k) := \sum_{i=1}^{k} f\big(2ax_i + \sum_{j=1, j\neq i}^{k} ax_j\big) - (k+2) f\big(\sum_{i=1}^{k} ax_i\big) - a \sum_{i=1}^{k} f(-x_i),$$

$$\Delta_a f(x_1, ..., x_k) := \sum_{i=1}^{k} f\big(2ax_i + \sum_{j=1, j\neq i}^{k} ax_j\big) - (k+2) f\big(\sum_{i=1}^{k} ax_i\big) - a^2 \sum_{i=1}^{k} f(-x_i),$$

$$M_a f(x_1, ..., x_k) := \sum_{i=1}^{k} f\big(2ax_i + \sum_{j=1, j\neq i}^{k} ax_j\big) - (k+2) f\big(\sum_{i=1}^{k} ax_i\big) - \frac{a^2 - a}{2} \sum_{i=1}^{k} f(-x_i) - \frac{a^2 + a}{2} \sum_{i=1}^{k} f(x_i),$$

for all $x_1, \ldots, x_k \in \mathbb{X}$ and $a \in B_1$.

Theorem 3.1. *Let $f : \mathbb{X} \to \mathbb{Y}$ be an odd function for which there exists a function $\varphi : \mathbb{X}^k \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{2^n} \varphi(2^n x_1, ..., 2^n x_k) = 0, \tag{3.1}$$

$$\|D_a f(x_1, ..., x_k)\| \leq \varphi(x_1, ..., x_k) \tag{3.2}$$

for all $x_1, \ldots, x_k \in \mathbb{X}$ *and all* $a \in B_1$*. If there exists a constant* $L < 1$ *such that*

$$\varphi(2x, 0, ..., 0) \leq 2L\varphi(x, 0, ..., 0)$$

for all $x \in \mathbb{X}$*, then there exists a unique additive function* $A : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\|f(x) - A(x)\| \leq \frac{1}{2-2L}\varphi(x, 0, ..., 0) \tag{3.3}$$

for all $x \in \mathbb{X}$*. Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathbb{X}$*, then* A *is* B*-linear.*

Proof. Replacing $x_1 = x$, $x_2 = ... = x_k = 0$ and $a = 1$ in (**??**), we get

$$\|f(2x) - 2f(x)\| \leq \varphi(x, 0, ..., 0) \tag{3.4}$$

for all $x \in \mathbb{X}$. Let E be the set of all odd functions $g : \mathbb{X} \to \mathbb{Y}$ and introduce a generalized metric on E as follows:

$$d(g, h) := \inf\{\, C \in [0, \infty] : \|g(x) - h(x)\| \leq C\varphi(x, 0, ..., 0)\,\}.$$

It is easy to show that (E, d) is a generalized complete metric space [**?**].

Now we consider the function $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = \frac{1}{2}g(2x), \quad \text{for all } g \in E \text{ and } x \in \mathbb{X}.$$

Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C$. From the definition of d, we have

$$\|g(x) - h(x)\| \leq C\varphi(x, 0, ..., 0)$$

for all $x \in X$. By the assumption and last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = \frac{1}{2}\|g(2x) - h(2x)\| \leq \frac{1}{2}C\varphi(2x, 0, ..., 0) \leq CL\varphi(x, 0, ..., 0)$$

for all $x \in \mathbb{X}$. So

$$d(\Lambda g, \Lambda h) \leq Ld(g, h)$$

for any $g, h \in E$. It follows from (**??**) that $d(\Lambda f, f) \leq \frac{1}{2}$. Therefore according to Theorem **??**, the sequence $\{\Lambda^n f\}$ converges to a fixed point A of Λ, i.e.,

$$A : \mathbb{X} \to \mathbb{Y}, \quad A(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

and $A(2x) = 2A(x)$ for all $x \in \mathbb{X}$. Also A is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f, g) < \infty\}$ and

$$d(A, f) \leq \frac{1}{1-L}d(\Lambda f, f) \leq \frac{1}{2-2L},$$

i.e., inequality (**??**) holds true for all $x \in \mathbb{X}$. It follows from the definition of A, (**??**) and (**??**) that

$$\begin{aligned}\|D_1A(x_1, ..., x_k)\| &= \lim_{n\to\infty} \frac{1}{2^n}\big\|D_1f(2^n x_1, ..., 2^n x_k)\big\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x_1, ..., 2^n x_k) = 0\end{aligned}$$

for all $x_1, ..., x_k \in \mathbb{X}$. Since f is odd, A is an odd function from the definition of A. By Lemma **??**, the function $A : \mathbb{X} \to \mathbb{Y}$ is additive. Finally it remains to prove the uniqueness of A. Let $T : \mathbb{X} \to \mathbb{Y}$ be another additive function satisfying (**??**). Since $d(f, T) \leq \frac{1}{2-2L}$ and T is additive, we get $T \in E^*$ and $(\Lambda T)(x) = \frac{1}{2}T(2x) = T(x)$ for all $x \in \mathbb{X}$, i.e., T is a fixed point of Λ. Since A is the unique fixed point of Λ in E^*, then $T = A$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then by the same reasoning as in the proof of [**?**] A is $\mathbb{R}$-linear.

Replacing $x_1 = x$ and $x_2 = ... = x_k = 0$ in (**??**), we get

$$\|f(2ax) + af(x) - 3f(ax)\| \leq \varphi(x, 0, ..., 0) \tag{3.5}$$

for all $x \in \mathbb{X}$ and all $a \in B_1$. So it follows from the definition of A, (**??**) and (**??**) that $A(2ax) + aA(x) = 3A(ax)$ for all $x \in \mathbb{X}$ and all $a \in B_1$. Since A is additive, $A(ax) = aA(x)$ for all $x \in \mathbb{X}$ and all $a \in B_1 \cup \{0\}$. Now, let $a \in B \setminus \{0\}$. Since A is $\mathbb{R}$-linear,

$$A(ax) = A\big(|a|.\frac{a}{|a|}x\big) = |a|A\big(\frac{a}{|a|}x\big) = |a|.\frac{a}{|a|}A(x) = aA(x)$$

for all $x \in \mathbb{X}$ and all $a \in B$. This proves that A is B-linear. □

Corollary 3.2. *Let $0 < r < 1$ and θ, δ be non-negative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be an odd function such that*

$$\|D_a f(x_1, ..., x_k)\| \leq \delta + \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in \mathbb{X}$ and all $a \in B_1$. Then there exists a unique additive function $A : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{\delta}{2 - 2^r} + \frac{\theta}{2 - 2^r}\|x\|^r$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then A is B-linear.

Proof. The proof follows from Theorem **??** by taking

$$\varphi(x_1, ..., x_k) := \delta + \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in \mathbb{X}$. Then we can choose $L = 2^{r-1}$ and we get the desired result. □

Theorem 3.3. *Let $f : \mathbb{X} \to \mathbb{Y}$ be an odd function for which there exists a function $\Phi : \mathbb{X}^k \to [0, \infty)$ such that*

$$\lim_{n\to\infty} 2^n \Phi\big(\frac{x_1}{2^n}, ..., \frac{x_k}{2^n}\big) = 0, \tag{3.6}$$

$$\|D_a f(x_1, ..., x_k)\| \leq \Phi(x_1, ..., x_k) \tag{3.7}$$

for all $x_1, \dots, x_k \in \mathbb{X}$ and all $a \in B_1$. If there exists a constant $L < 1$ such that

$$2\Phi(x, 0, ..., 0) \leq L\Phi(2x, 0, ..., 0)$$

for all $x \in \mathbb{X}$, then there exists a unique additive function $A : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{L}{2 - 2L}\Phi(x, 0, ..., 0) \tag{3.8}$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then A is B-linear.

Proof. Using the same method as in the proof of Theorem **??**, we have

$$\Big\|2f\big(\frac{x}{2}\big) - f(x)\Big\| \leq \Phi\big(\frac{x}{2}, 0, ..., 0\big) \leq \frac{L}{2}\Phi(x, 0, ..., 0) \tag{3.9}$$

for all $x \in \mathbb{X}$. We introduce the same definitions for E and d as in the proof of Theorem **??** such that (E, d) becomes a generalized complete metric space (replacing φ by Φ). Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda g)(x) = 2g\big(\frac{x}{2}\big), \quad \text{for all } g \in E \text{ and } x \in X.$$

One can show that $d(\Lambda g, \Lambda h) \leq Ld(g, h)$ for any $g, h \in E$. It follows from (**??**) that $d(\Lambda f, f) \leq \frac{L}{2}$. Due to Theorem **??**, the sequence $\{\Lambda^n f\}$ converges to a fixed point A of Λ, i.e.,

$$A : \mathbb{X} \to \mathbb{Y}, \quad A(x) = \lim_{n\to\infty}(\Lambda^n f)(x) = \lim_{n\to\infty} 2^n f\big(\frac{x}{2^n}\big)$$

and $A(2x) = 2A(x)$ for all $x \in \mathbb{X}$. Also

$$d(A, f) \leq \frac{1}{1 - L}d(\Lambda f, f) \leq \frac{L}{2 - 2L},$$

i.e., inequality (**??**) holds true for all $x \in \mathbb{X}$.

The rest of the proof is similar to the proof of Theorem **??** and we omit the details. □

Corollary 3.4. *Let $r > 1$ and θ be non-negative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be an odd function such that*

$$\|D_a f(x_1, ..., x_k)\| \leq \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \dots, x_k \in \mathbb{X}$ and all $a \in B_1$. Then there exists a unique additive function $A : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{\theta}{2^r - 2}\|x\|^r$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then A is B-linear.

Proof. The proof follows from Theorem **??** by taking

$$\Phi(x_1, ..., x_k) := \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \dots, x_k \in \mathbb{X}$. Then we can choose $L = 2^{1-r}$ and we get the desired result. □

Theorem 3.5. *Let $f : \mathbb{X} \to \mathbb{Y}$ be an even function with $f(0) = 0$ for which there exists a function $\psi : \mathbb{X}^k \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{4^n} \psi(2^n x_1, ..., 2^n x_k) = 0, \tag{3.10}$$

$$\|\Delta_a f(x_1, ..., x_k)\| \leq \psi(x_1, ..., x_k) \tag{3.11}$$

for all $x_1, \dots, x_k \in \mathbb{X}$ and all $a \in B_1$. If there exists a constant $L < 1$ such that

$$\psi(2x, 0, ..., 0) \leq 4L\psi(x, 0, ..., 0)$$

for all $x \in \mathbb{X}$, then there exists a unique quadratic function $Q : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{4 - 4L} \psi(x, 0, ..., 0) \tag{3.12}$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then Q is B-quadratic.

Proof. Similar to the proof of Theorem **??**, we have

$$\|f(2x) - 4f(x)\| \leq \psi(x, 0, ..., 0) \tag{3.13}$$

for all $x \in \mathbb{X}$. Let E be the set of all even functions $g : \mathbb{X} \to \mathbb{Y}$ with $g(0) = 0$ and introduce a generalized metric on E as follows:

$$d(g, h) := \inf\{\, C \in [0, \infty] : \|g(x) - h(x)\| \leq C\psi(x, 0, ..., 0) \,\}.$$

So (E, d) is a generalized complete metric space. Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda g)(x) = \frac{1}{4} g(2x), \quad \text{for all } g \in E \text{ and } x \in X.$$

One can show that $d(\Lambda g, \Lambda h) \leq Ld(g, h)$ for any $g, h \in E$. It follows from (**??**) that $d(\Lambda f, f) \leq \frac{1}{4}$. Due to Theorem **??**, the sequence $\{\Lambda^n f\}$ converges to a fixed point Q of Λ, i.e.,

$$Q : \mathbb{X} \to \mathbb{Y}, \quad Q(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

and $Q(2x) = 4Q(x)$ for all $x \in \mathbb{X}$. Also Q is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f, g) < \infty\}$ and

$$d(Q, f) \leq \frac{1}{1 - L} d(\Lambda f, f) \leq \frac{1}{4 - 4L},$$

i.e., inequality (**??**) holds true for all $x \in \mathbb{X}$. It follows from the definition of Q, (**??**) and (**??**) that

$$\begin{aligned}\|\Delta_1 Q(x_1, ..., x_k)\| &= \lim_{n\to\infty} \frac{1}{4^n}\big\|\Delta_1 f(2^n x_1, ..., 2^n x_k)\big\| \\ &\leq \lim_{n\to\infty} \frac{1}{2^n}\psi(2^n x_1, ..., 2^n x_k) = 0\end{aligned}$$

for all $x_1, ..., x_k \in \mathbb{X}$. Since f is even, Q is an even function from the definition of Q. By Lemma **??**, the function $Q : \mathbb{X} \to \mathbb{Y}$ is quadratic.

Finally it remains to prove the uniqueness of Q. Let $T : \mathbb{X} \to \mathbb{Y}$ be another quadratic function satisfying (**??**). Since $d(f, T) \leq \frac{1}{4-4L}$, and T is quadratic, then $T \in E^*$ and $(\Lambda T)(x) = \frac{1}{4}T(2x) = T(x)$ for all $x \in \mathbb{X}$, i.e., T is a fixed point of Λ. Since Q is the unique fixed point of Λ in E^*, then $T = Q$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then by the same reasoning as in the proof of [**?**] Q is $\mathbb{R}$-quadratic.

Replacing $x_1 = x$ and $x_2 = ... = x_k = 0$ in (**??**), we get

$$\|f(2ax) - 3f(ax) - a^2 f(x)\| \leq \psi(x, 0, ..., 0) \tag{3.14}$$

for all $x \in \mathbb{X}$ and all $a \in B_1$. So it follows from the definition of A, (**??**) and (**??**) that $Q(2ax) - 3Q(ax) = a^2 Q(x)$ for all $x \in \mathbb{X}$ and all $a \in B_1$. Since Q is quadratic, $Q(ax) = a^2 Q(x)$ for all $x \in \mathbb{X}$ and all $a \in B_1 \cup \{0\}$. Now, let $a \in B \setminus \{0\}$. Since Q is $\mathbb{R}$-quadratic,

$$Q(ax) = Q\big(|a|.\frac{a}{|a|}x\big) = |a|^2 Q\big(\frac{a}{|a|}x\big) = |a|^2.\frac{a^2}{|a|^2}Q(x) = a^2 Q(x)$$

for all $x \in \mathbb{X}$ and all $a \in B$. This proves that Q is B-quadratic. □

Corollary 3.6. *Let $0 < r < 2$ and θ, δ be non-negative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be an even function with $f(0) = 0$ such that*

$$\|\Delta_a f(x_1, ..., x_k)\| \leq \delta + \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in \mathbb{X}$ and all $a \in B_1$. Then there exists a unique quadratic function $A : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{\delta}{4 - 2^r} + \frac{\theta}{4 - 2^r}\|x\|^r$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then Q is B-quadratic.

Proof. The proof follows from Theorem **??** by taking

$$\psi(x_1, ..., x_k) := \delta + \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in \mathbb{X}$. Then we can choose $L = 2^{r-2}$ and we get the desired result. □

Theorem 3.7. *Let $f : \mathbb{X} \to \mathbb{Y}$ be an even function for which there exists a function $\Psi : \mathbb{X}^k \to [0, \infty)$ such that*

$$\lim_{n\to\infty} 4^n \Psi\big(\frac{x_1}{2^n}, ..., \frac{x_k}{2^n}\big) = 0,$$
$$\|\Delta_a f(x_1, ..., x_k)\| \le \Psi(x_1, ..., x_k) \tag{3.15}$$

for all $x_1, \dots, x_k \in \mathbb{X}$ and all $a \in B_1$. If there exists a constant $L < 1$ such that

$$4\Psi(x, 0, ..., 0) \le L\Psi(2x, 0, ..., 0)$$

for all $x \in \mathbb{X}$, then there exists a unique quadratic function $Q : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - Q(x)\| \le \frac{L}{4 - 4L}\Psi(x, 0, ..., 0)$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then Q is B-quadratic.

Proof. It follows from (**??**) that $f(0) = 0$. Using the same method as in the proof of Theorem **??**, we have

$$\Big\|4f\big(\frac{x}{2}\big) - f(x)\Big\| \le \Psi\big(\frac{x}{2}, 0, ..., 0\big) \le \frac{L}{4}\Psi(x, 0, ..., 0) \tag{3.16}$$

for all $x \in \mathbb{X}$. We introduce the same definitions for E and d as in the proof of Theorem **??** such that (E, d) becomes a generalized complete metric space (replacing ψ by Ψ). Let $\Lambda : E \to E$ be the mapping defined by

$$(\Lambda g)(x) = 4g\big(\frac{x}{2}\big), \quad \text{for all } g \in E \text{ and } x \in X.$$

One can show that $d(\Lambda g, \Lambda h) \le Ld(g, h)$ for any $g, h \in E$. It follows from (**??**) that $d(\Lambda f, f) \le \frac{L}{4}$. Due to Theorem **??**, the sequence $\{\Lambda^n f\}$ converges to a fixed point Q of Λ, i.e.,

$$Q : \mathbb{X} \to \mathbb{Y}, \quad Q(x) = \lim_{n\to\infty} (\Lambda^n f)(x) = \lim_{n\to\infty} 4^n f\big(\frac{x}{2^n}\big)$$

and $Q(2x) = 4Q(x)$ for all $x \in \mathbb{X}$. Also

$$d(Q, f) \le \frac{1}{1 - L} d(\Lambda f, f) \le \frac{L}{4 - 4L},$$

i.e.,

$$\|f(x) - Q(x)\| \le \frac{L}{4 - 4L}\Psi(x, 0, ..., 0)$$

holds true for all $x \in \mathbb{X}$.

The rest of the proof is similar to the proof of Theorem **??** and we omit the details. □

Corollary 3.8. *Let $r > 2$ and θ be non-negative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be an even function such that*

$$\|\Delta_a f(x_1, ..., x_k)\| \le \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in \mathbb{X}$ and all $a \in B_1$. Then there exists a unique quadratic function $Q : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{2^r - 4}\|x\|^r$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then Q is B-quadratic.

Proof. The proof follows from Theorem **??** by taking

$$\Psi(x_1, ..., x_k) := \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in \mathbb{X}$. Then we can choose $L = 2^{2-r}$ and we get the desired result. □

Theorem 3.9. *Let $f : \mathbb{X} \to \mathbb{Y}$ be a function with $f(0) = 0$ for which there exists a function $\varphi : \mathbb{X}^k \to [0, \infty)$ satisfying (**??**) and*

$$\|M_a f(x_1, ..., x_k)\| \leq \varphi(x_1, ..., x_k) \tag{3.17}$$

for all $x_1, \ldots, x_k \in \mathbb{X}$ and all $a \in B_1$. If there exists a constant $L < 1$ such that

$$\varphi(2x, 0, ..., 0) \leq 2L\varphi(x, 0, ..., 0)$$

for all $x \in \mathbb{X}$, then there exist a unique additive function $A : \mathbb{X} \to \mathbb{Y}$ and a unique quadratic function $Q : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \frac{3}{8 - 8L}[\varphi(x, 0, ..., 0) + \varphi(-x, 0, ..., 0)] \tag{3.18}$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then A is B-linear and Q is B-quadratic.

Proof. We decompose f into the even part and the odd part by putting

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in \mathbb{X}$. It is clear that $f(x) = f_e(x) + f_o(x)$ for all $x \in \mathbb{X}$. Let

$$\psi(x_1, ..., x_k) := \frac{1}{2}[\varphi(x_1, ..., x_k) + \varphi(-x_1, ..., -x_k)]$$

for all $x_1, \ldots, x_k \in \mathbb{X}$. So we have from (**??**) and (**??**) that

$$\lim_{n\to\infty} \frac{1}{2^n}\psi(2^n x_1, ..., 2^n x_k) = 0,$$
$$\|M_a f_e(x_1, ..., x_k)\| = \|\Delta_a f_e(x_1, ..., x_k)\| \leq \psi(x_1, ..., x_k),$$
$$\|M_a f_o(x_1, ..., x_k)\| = \|D_a f_o(x_1, ..., x_k)\| \leq \psi(x_1, ..., x_k)$$

and

$$\psi(2x, 0, ..., 0) \leq 2L\psi(x, 0, ..., 0)$$

for all $x_1, \ldots, x_k \in \mathbb{X}$ and $a \in B_1$. Hence the result follows from Theorems **??** and **??**. □

Corollary 3.10. *Let $0 < r < 1$ and θ, δ be non-negative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be a function with $f(0) = 0$ such that*

$$\|M_a f(x_1, ..., x_k)\| \leq \delta + \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \dots, x_k \in \mathbb{X}$ and all $a \in B_1$. Then there exist a unique additive function $A : \mathbb{X} \to \mathbb{Y}$ and a unique quadratic function $Q : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \frac{3\delta}{2(2-2^r)} + \frac{3\theta}{2(2-2^r)}\|x\|^r$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then A is B-linear and Q is B-quadratic.

Proof. The proof follows from Theorem **??** by taking

$$\varphi(x_1, ..., x_k) := \delta + \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \dots, x_k \in \mathbb{X}$. Then we can choose $L = 2^{r-1}$ and we get the desired result. □

Theorem 3.11. *Let $f : \mathbb{X} \to \mathbb{Y}$ be a function for which there exists a function $\varphi : \mathbb{X}^k \to [0, \infty)$ satisfying*

$$\lim_{n\to\infty} 4^n \varphi\big(\frac{x_1}{2^n}, ..., \frac{x_k}{2^n}\big) = 0,$$
$$\|M_a f(x_1, ..., x_k)\| \leq \varphi(x_1, ..., x_k)$$

for all $x_1, \dots, x_k \in \mathbb{X}$ and all $a \in B_1$. If there exists a constant $L < 1$ such that

$$4\varphi(x, 0, ..., 0) \leq L\varphi(2x, 0, ..., 0)$$

for all $x \in \mathbb{X}$, then there exist a unique additive function $A : \mathbb{X} \to \mathbb{Y}$ and a unique quadratic function $Q : \mathbb{X} \to \mathbb{Y}$ such that

$$\|f(x) - A(x) - Q(x)\| \leq \frac{3L}{8-8L}[\varphi(x, 0, ..., 0) + \varphi(-x, 0, ..., 0)]$$

for all $x \in \mathbb{X}$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{X}$, then A is B-linear and Q is B-quadratic.

Proof. Similar to the proof of Theorem **??**, the result follows from Theorems **??** and **??**. □

Corollary 3.12. *Let $r > 2$ and θ be non-negative real numbers and let $f : \mathbb{X} \to \mathbb{Y}$ be a function such that*

$$\|M_a f(x_1, ..., x_k)\| \leq \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in \mathbb{X}$ *and all* $a \in B_1$*. Then there exist a unique additive function* $A : \mathbb{X} \to \mathbb{Y}$ *and a unique quadratic function* $Q : \mathbb{X} \to \mathbb{Y}$ *such that*

$$\|f(x) - A(x) - Q(x)\| \leq \frac{3\theta}{2^r - 4}\|x\|^r$$

for all $x \in \mathbb{X}$*. Moreover, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in \mathbb{X}$*, then* A *is* B*-linear and* Q *is* B*-quadratic.*

Proof. The proof follows from Theorem **??** by taking

$$\varphi(x_1, ..., x_k) := \theta \sum_{i=1}^{k} \|x_i\|^r$$

for all $x_1, \ldots, x_k \in \mathbb{X}$. Then we can choose $L = 2^{2-r}$ and we get the desired result. □

Acknowledgment

The authors would like to thank the referee for a number of valuable suggestions regarding a previous version of this paper.

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