Stability and superstability of Jordan homomorphisms and Jordan derivations on Banach algebras and C^*-algebras: A fixed point approach

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Abstract. Using fixed point methods, we prove the Hyers–Ulam–Rassias stability and superstability of Jordan homomorphisms (Jordan $*$-homomorphisms), and Jordan derivations (Jordan $*$-derivations) on Banach algebras (C^*-algebras) for the generalized Jensen–type functional equation

$$rf\Big(\frac{x+y}{r}\Big)+rf\Big(\frac{x-y}{r}\Big)=2f(x)$$

where r is a fixed positive real number in $(1,\infty)$.

1. Introduction

A classical result of Herstein [?] asserts that any Jordan derivation on a prime ring with characteristic different from two is a derivation. A brief proof of Herstein's result can be found in 1988 by Brear and Vukman [?]. Cusack [?] generalized Herstein's result to 2-torsion-free semiprime rings (see also [?] for an alternative proof). For some other results concerning derivations on prime and semiprime rings, we refer to [?, ?, ?].

We say a functional equation (ξ) is stable if any function g satisfying the equation (ξ) *approximately* is near to a true solution of (ξ). We say that a functional equation is *superstable* if every approximately solution is an exact solution of it.

The stability of functional equations was first introduced by S. M. Ulam [?] in 1940. More precisely, he proposed the following problem: Given a group G_1, a metric group (G_2, d) and a positive number ϵ, does there exist a $\delta > 0$ such that if a function

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$f : G_1 \longrightarrow G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$? As mentioned above, when this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, D.H. Hyers [?] gave a partial solution of *Ulam's* problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th.M. Rassias [?] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. This phenomenon of stability that was introduced by Th.M. Rassias [?] is called the *Hyers–Ulam–Rassias stability* of functional equations.

Theorem 1.1. [?] *Let $f : E \longrightarrow E'$ be a mapping from a norm vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \longrightarrow E'$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in E$. If $p < 0$ then inequality (??) holds for all $x, y \neq 0$, and (??) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into E' is continuous for each fixed $x \in E$, then T is linear.

The result of the Rassias theorem has been generalized by Forti [?] and Găvruta [?] who permitted the Cauchy difference to become arbitrary unbounded. Some results on the stability of functional equations in single variable and nonlinear iterative equations can be found in [?, ?].

During the last decades several stability problems of functional equations have been investigated by many mathematicians. A large list of references concerning the stability of functional equations can be found in [?, ?, ?].

D.G. Bourgin is the first mathematician dealing with the stability of ring homomorphisms. The topic of approximate ring homomorphisms was studied by a number of mathematicians, see [?, ?, ?, ?, ?, ?, ?, ?, ?] and references therein.

It seems that approximate derivations was first investigated by Jun and D. Park[?]. Recently, the stability of derivations has been investigated by some authors; see [?, ?] and references therein.

Miura al et. [?] proved the Hyers–Ulam–Rassias stability of Jordan homomorphisms, also, Jun and H. Kim [?] proved the Hyers–Ulam–Rassias stability of Jordan derivations on Banach algebras.

Recently, Cădariu and Radu applied the fixed point method to the investigation of the functional equations. (see also [?, ?, ?, ?, ?, ?, ?]). Before proceeding to the main results, we will state the following theorem.

Theorem 1.2. *(The alternative of fixed point [?]). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$d(T^m x, T^{m+1} x) = \infty$ *for all* $m \ge 0$,

or other exists a natural number m_0 such that

- ⋆ *$d(T^m x, T^{m+1}x) < \infty$ for all $m \geq m_0$;*
- ⋆ *the sequence $\{T^m x\}$ is convergent to a fixed point y^* of T;*
- ⋆ *y^* is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{m_0}x, y) < \infty\}$;*
- ⋆ *$d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Lambda$.*

In this paper, we will adopt the fixed point alternative of Cădariu and Radu to prove the Hyers–Ulam–Rassias stability and superstability of Jordan homomorphisms (Jordan $*$-homomorphisms) and Jordan derivations (Jordan $*$-derivations) on Banach algebras (C^*-algebras) associated with the following generalized Jensen–type functional equation

$$rf\Big(\frac{x+y}{r}\Big) + rf\Big(\frac{x-y}{r}\Big) = 2f(x)$$

where r is a fixed positive real number in $(1, \infty)$.

Throughout this paper assume that A, B are two Banach algebras. For a given mapping $f : A \to B$, we define

$$\Delta_\mu f(x, y) = r\mu f\Big(\frac{x+y}{r}\Big) + r\mu f\Big(\frac{x-y}{r}\Big) - 2f(\mu x)$$

for all $\mu \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ and all $x, y \in A$.

2. Jordan homomorphisms and Jordan derivations on Banach algebras

Note that a $\mathbb{C}$-linear mapping $h : A \to B (D : A \to A)$ is a Jordan homomorphism (Jordan derivation) if $h(D)$ satisfies

$$h(a^2) = (h(a))^2, \qquad (D(a^2) = aD(a) + D(a)a)$$

for all $a \in A$.

Lemma 2.1. [?] *Let $f : X \to Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}$, where X and Y are linear spaces. Then the mapping f is $\mathbb{C}$-linear.*

Theorem 2.2. *Let $f : A \to B$ be a mapping for which there exists a function $\phi : A^2 \to [0, \infty)$ such that*

$$\|\Delta_\mu f(x, y)\| \leq \phi(x, y), \tag{2.1}$$

$$\|f(x^2) - (f(x))^2\| \leq \phi(x, y) \tag{2.2}$$

for all $\mu \in \mathbb{T}$ and all $x \in A$. If there exists an $L < 1$ such that $\phi(x, y) \leq rL\phi(\frac{x}{r}, \frac{y}{r})$ for all $x, y \in A$, then there exists a unique Jordan homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\| \leq \frac{L}{1-L}\phi(x, 0) \tag{2.3}$$

for all $x \in A$.

Proof. It follows from $\phi(x, y) \leq rL\phi(\frac{x}{r}, \frac{y}{r})$ that

$$\lim_j r^{-j}\phi(r^j x, r^j y) = 0 \tag{2.4}$$

for all $x, y \in A$. Putting $\mu = 1, y = 0$ in (**??**), we get

$$\|rf(\frac{x}{r}) - f(x)\| \leq \phi(x, 0) \tag{2.5}$$

for all $x \in A$. Hence,

$$\|\frac{1}{r}f(rx) - f(x)\| \leq \frac{1}{r}\phi(rx, 0) \leq L\phi(x, 0) \tag{2.6}$$

for all $x \in A$. Consider the set $X := \{g \mid g : A \to B\}$ and introduce the generalized metric on X:

$$d(h, g) := \inf\{C \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq C\phi(x, 0) \text{ for all } x \in A\}.$$

It is easy to show that (X, d) is complete. Now we define the linear mapping $J : X \to X$ by

$$J(h)(x) = \frac{1}{r}h(rx)$$

for all $x \in A$. By [**??**, Theorem 3.1],

$$d(J(g), J(h)) \leq Ld(g, h)$$

for all $g, h \in X$. It follows from (**??**) that

$$d(f, J(f)) \leq L.$$

By Theorem **??**, J has a unique fixed point in the set $X_1 := \{h \in X : d(f, h) < \infty\}$. Let h be the fixed point of J. Then h is the unique mapping with

$$h(rx) = rh(x)$$

for all $x \in A$. On the other hand we have $\lim_n d(J^n(f), h) = 0$. So that

$$\lim_n \frac{1}{r^n}f(r^n x) = h(x) \tag{2.7}$$

for all $x \in A$. It follows from $d(f, h) \leq \frac{1}{1-L}d(f, J(f))$ that

$$d(f, h) \leq \frac{L}{1 - L}.$$

This implies the inequality (**??**). It follows from (**??**), (**??**) and (**??**) that

$$\begin{aligned}
&\Big\|rh(\frac{x+y}{r}) + rh(\frac{x-y}{r}) - 2h(x)\Big\| \\
&= \lim_n \frac{1}{r^n}\|f(r^{n-1}(x + y)) + f(r^{n-1}(x - y)) - f(r^n x)\| \\
&\leq \lim_n \frac{1}{r^n}\phi(r^n x, r^n y) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$rh(\frac{x+y}{r}) + rh(\frac{x-y}{r}) = 2h(x)$$

for all $x, y \in A$. Putting $z = \frac{x+y}{r}$ and $t = \frac{x-y}{r}$ in the above equation, we get

$$h(z) + h(t) = \frac{2}{r}h\Big(\frac{r(z + t)}{2}\Big) \tag{2.8}$$

for all $z, t \in A$. Putting $t = 0$ in the above equation, we get $\frac{r}{2}h(z) = h(\frac{r}{2}z)$. So, $\frac{r}{2}h(t) = h(\frac{r}{2}t)$. Hence, H is Cauchy additive by using (**??**). By putting $y = x$ in (**??**), we have

$$\|r\mu f(\frac{2x}{r}) - 2f(\mu x)\| \leq \phi(x, x)$$

for all $x \in A$. This implies that

$$\|h(2\mu x) - 2\mu h(x)\| = \lim_n \frac{1}{r^n}\|f(2\mu r^n x) - 2\mu f(r^n x)\| \leq \lim_n \frac{1}{r^n}\phi(r^n x, r^n x) = 0$$

for all $\mu \in \mathbb{T}$ and all $x \in A$. By Lemma **??**, the mapping $h : A \to B$ is $\mathbb{C}$-linear. Since $r > 1$, it follows from (**??**) that

$$\begin{aligned}
&\|h(x^2) - (h(x))^2\| \\
&= \lim_n \left\| \frac{1}{r^{2n}}h(x^2) - \frac{1}{r^{2n}}(h(x))^2 \right\| \\
&\leq \lim_n \frac{1}{r^{2n}}\phi(r^n x, r^n x) \leq \lim_n \frac{1}{r^n}\phi(r^n x, r^n x) \\
&= 0
\end{aligned}$$

for all $x \in A$. Thus $h : A \to B$ is a Jordan homomorphism satisfying (**??**), as desired. □

We prove the following Hyers–Ulam–Rassias stability problem for Jordan homomorphisms on Banach algebras.

Corollary 2.3. *Let $p \in (0, 1)$ and $\theta \in [0, \infty)$ be real numbers. Suppose $f : A \to B$ satisfies*

$$\|\Delta_\mu f(x, y)\| \leq \theta(\|x\|^p + \|y\|^p),$$
$$\|f(x^2) - (f(x))^2\| \leq 2\theta\|x\|^p$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A$. Then there exists a unique Jordan homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\| \leq \frac{2^p\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. Set $\phi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in A$, and let $L = 2^{p-1}$ in Theorem **??**. Then we get the desired result. □

Theorem 2.4. *Let $f : A \to A$ be a mapping for which there exists a function $\phi : A^2 \to [0, \infty)$ satisfying (**??**) and*

$$\|f(x^2) - xf(x) - f(x)x\| \leq \phi(x, y) \tag{2.9}$$

for all $x \in A$. If there exists an $L < 1$ such that $\phi(x, y) \leq rL\phi(\frac{x}{r}, \frac{y}{r})$ for all $x, y \in A$, then there exists a unique Jordan derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \leq \frac{L}{1 - L}\phi(x, 0) \tag{2.10}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem **??**, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (**??**). The mapping $D : A \to A$ is given by

$$D(x) = \lim_n \frac{1}{r^n} f(r^n x)$$

for all $x \in A$. It follows from (**??**) that

$$\begin{aligned}
&\|D(x^2) - xD(x) - D(x)x\| \\
&\quad = \lim_n \left\| \frac{1}{r^{2n}} D(x^2) - \frac{1}{r^{2n}} (D(x)x - xD(x)) \right\| \\
&\quad \le \lim_n \frac{1}{r^{2n}} \phi(r^n x, r^n x) \le \lim_n \frac{1}{r^n} \phi(r^n x, r^n x) \\
&\quad = 0
\end{aligned}$$

for all $x \in A$. Thus $D : A \to A$ is a Jordan derivation satisfying (**??**). □

We prove the following Hyers–Ulam–Rassias stability problem for Jordan derivation on Banach algebras.

Corollary 2.5. *Let $p \in (0, 1)$ and $\theta \in [0, \infty)$ be real numbers. Suppose $f : A \to A$ satisfies*

$$\|\Delta_\mu f(x, y)\| \le \theta(\|x\|^p + \|y\|^p),$$
$$\|f(x^2) - xf(x) - f(x)x\| \le 2\theta\|x\|^p$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A$. Then there exists a unique Jordan derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \le \frac{2^p \theta}{2 - 2^p} \|x\|^p$$

for all $x \in A$.

Proof. Set $\phi(x, y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in A$, and let $L = 2^{p-1}$ in Theorem **??**. Then we get the desired result. □

The case in which $p = 1$ was excluded in Corollary **??** and Corollary **??**. Indeed these corollaries are not valid when $p = 1$. Here we use the Gajda's example [**?**] to give a counter example.

Proposition 2.6. *Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by*

$$\phi(x) := \begin{cases} x & \text{for } |x| < 1; \\ 1 & \text{for } |x| \ge 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} 2^{-n} \phi(2^n x).$$

Then f satisfies

$$\big|\Delta_\mu f(x,y)\big| \leq 8(r+1)(|x|+|y|), \tag{2.11}$$
$$|f(x^2)-(f(x))^2| \leq 12|x|, \tag{2.12}$$
$$|f(x^2)-2xf(x)| \leq 8|x| \tag{2.13}$$

for all $\mu \in \mathbb{T}$ and all $x, y \in \mathbb{C}$, and the range of $|f(x)-A(x)|/|x|$ for $x \neq 0$ is unbounded for each additive mapping $A : \mathbb{C} \to \mathbb{C}$.

Proof. It is clear that f is bounded by 2 on $\mathbb{C}$. If $|x|+|y| = 0$ or $|x|+|y| \geq \frac{1}{2}$, then

$$\big|\Delta_\mu f(x,y)\big| \leq 8(r+1)(|x|+|y|).$$

Now suppose that $|x|+|y| < \frac{1}{2}$. Then there exists an integer $k \geq 1$ such that

$$\frac{1}{2^{k+1}} \leq |x|+|y| < \frac{1}{2^k}. \tag{2.14}$$

Therefore

$$2^m r^{-1}|x \pm y|, 2^m|x| < 1$$

for all $m = 0, 1, ..., k-1$. From the definition of f and (**??**), we have

$$\begin{aligned}\big|\Delta_\mu f(x,y)\big| &\leq \sum_{n=k}^{\infty} 2^{-n}\Big[r|\phi(2^n r^{-1}(x+y))| + r|\phi(2^n r^{-1}(x-y))| + 2|\phi(2^n \mu x)|\Big]\\ &\leq \frac{4}{2^k}(r+1) \leq 8(r+1)(|x|+|y|).\end{aligned}$$

Therefore f satisfies (**??**). To prove (**??**) and (**??**), if $|x| = 0$ or $|x| \geq \frac{1}{2}$, then

$$\begin{aligned}|f(x^2)-(f(x))^2| &\leq 6 \leq 12|x|,\\ |f(x^2)-2xf(x)| &\leq 2+4|x| \leq 8|x|.\end{aligned}$$

For $0 < |x| < \frac{1}{2}$, we choose an integer $k \geq 1$ such that $\frac{1}{2^{k+1}} \leq |x| < \frac{1}{2^k}$. Therefore

$$2^m|x|^2, 2^m|x| < 1$$

for all $m = 0, 1, ..., k-1$. So

$$\begin{aligned}|f(x^2)-(f(x))^2| &= \Big|(k-k^2)x^2 + \sum_{n=k}^{\infty} 2^{-n}\phi(2^n x^2) - 2kx\sum_{n=k}^{\infty} 2^{-n}\phi(2^n x) - \big(\sum_{n=k}^{\infty} 2^{-n}\phi(2^n x)\big)^2\Big|\\ &\leq \frac{k^2-k}{2^k}|x| + \frac{2}{2^k} + \frac{4k}{2^k}|x| + \big(\frac{2}{2^k}\big)^2\\ &\leq \frac{k^2+3k}{2^k}|x| + 8|x| \leq 12|x|,\end{aligned}$$

also

$$|f(x^2)-2xf(x)| = \Big| -kx^2+\sum_{n=k}^{\infty}2^{-n}\phi(2^nx^2)-2x\sum_{n=k}^{\infty}2^{-n}\phi(2^nx)\Big|$$
$$\leq k|x|^2+\frac{2}{2^k}+\frac{4}{2^k}|x|$$
$$\leq \frac{k+4}{2^k}|x|+4|x|\leq 7|x|$$

Hence f satisfies (**??**) and (**??**). Let $A:\mathbb{C}\to\mathbb{C}$ be an additive function such that

$$|f(x)-A(x)|\leq \beta|x|$$

for all $x\in\mathbb{C}$. Then there exists a constant $c\in\mathbb{R}$ such that $A(x)=cx$ for all rational numbers x. So we have

$$|f(x)|\leq(\beta+|c|)|x| \tag{2.15}$$

for all rational numbers x. Let $m\in\mathbb{N}$ with $m>\beta+|c|$. If x is a rational number in $(0,2^{1-m})$, then $2^nx\in(0,1)$ for all $n=0,1,...,m-1$. So

$$f(x)\geq\sum_{n=0}^{m-1}2^{-n}\phi(2^nx)=mx>(\beta+|c|)x$$

which contradicts (**??**). □

3. Jordan ∗-homomorphisms and Jordan ∗-derivations on C^*-algebras

Throughout this section assume that A and B are two C^*-algebras. Note that a Jordan homomorphism $h:A\to B$ $(D:A\to A)$ is an ∗-Jordan homomorphism (∗-Jordan derivation) if $h(D)$ satisfies

$$h(a^*)=(h(a))^*,\qquad (D(a^*)=(D(a))^*)$$

for all $a\in A$.

We establish the Hyers–Ulam–Rassias stability of ∗-Jordan homomorphisms and ∗-Jordan derivations in C^*-algebras by using the alternative fixed point theorem.

Theorem 3.1. *Let $f:A\to B$ be a mapping for which there exists a function $\phi: A^2\to[0,\infty)$ satisfying (**??**), (**??**) and*

$$\|f(x^*)-(f(x))^*\|\leq\phi(x,y) \tag{3.1}$$

for all $\mu\in\mathbb{T}$ and all $x\in A$. If there exists $0<L<1$ such that $\phi(x,y)\leq rL\phi(\frac{x}{r},\frac{y}{r})$ for all $x,y\in A$, then there exists a unique Jordan ∗-homomorphism $h:A\to B$ such that

$$\|f(x)-h(x)\|\leq\frac{L}{1-L}\phi(x,0) \tag{3.2}$$

for all $x\in A$.

Proof. By the same reasoning as in the proof of Theorem **??**, there exists a unique Jordan homomorphism $h : A \to B$ satisfying (**??**) and h is given by

$$h(x) = \lim_n \frac{1}{r^n} f(r^n x)$$

for all $x \in A$. We have to show that h is $*$-preserving. To this end, it follows from (**??**) that

$$\begin{aligned}&\|h(x^*) - (h(x))^*\| \\ &= \lim_n \left\| \frac{1}{r^{2n}} h(x^*) - \frac{1}{r^{2n}} (h(x))^* \right\| \\ &\leq \lim_n \frac{1}{r^{2n}} \phi(r^n x, r^n x) \leq \lim_n \frac{1}{r^n} \phi(r^n x, r^n x) \\ &= 0\end{aligned}$$

for all $x \in A$. Thus $h : A \to B$ is a Jordan $*$-homomorphism satisfying (**??**), as desired. □

We prove the following Hyers–Ulam–Rassias stability problem for Jordan $*$-homomorphisms on C^*-algebras.

Corollary 3.2. *Let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers. Suppose $f : A \to B$ satisfies*

$$\begin{aligned}\max\{\|f(x^*) - (f(x))^*\|, \|f(x^2) - (f(x))^2\|\} \leq 2\theta\|x\|^p, \\ \|\Delta_\mu f(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)\end{aligned}$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A$. So Then there exists a unique Jordan $*$-homomorphism $h : A \to B$ such that

$$\|f(x) - h(x)\| \leq \frac{2^p\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. Set $\phi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in A$, and let $L = 2^{p-1}$ in Theorem **??**. Then we get the desired result.

□

In 1996, Johnson [**?**] proved the following theorem (see also [**??**, Theorem 2.4]).

Theorem 3.3. *Suppose $\mathcal{A}$ is a C^*-algebra and $\mathcal{M}$ is a Banach $\mathcal{A}-$module. Then each Jordan derivation $d : \mathcal{A} \to \mathcal{M}$ is a derivation.*

Theorem 3.4. *Let $f : A \to A$ be a mapping for which there exists a function $\phi : A^2 \to [0,\infty)$ satisfying (**??**), (**??**) and (**??**). If there exists $0 < L < 1$ such that $\phi(x,y) \leq rL\phi(\frac{x}{r}, \frac{y}{r})$ for all $x, y \in A$, then there exists a unique $*$-derivation $D : A \to A$ such that*

$$\|f(x) - D(x)\| \leq \frac{L}{1-L}\phi(x,0) \tag{3.3}$$

for all $x \in A$.

Proof. Similarly to the proofs of Theorems **??** and **??**, we can show that there exists a unique Jordan $*$-derivation $D : A \to A$ satisfying (**??**). By Theorem **??**, D is a derivation, and the proof is complete. □

We prove the following Hyers–Ulam–Rassias stability problem for Jordan $*$-derivations on Banach algebras.

Corollary 3.5. *Let $p \in (0,1)$ and $\theta \in [0,\infty)$ be real numbers. Suppose $f : A \to A$ satisfies*

$$\max\{\|f(x^*) - (f(x))^*\|, \|f(x^2) - xf(x) - f(x)x\|\} \leq 2\theta\|x\|^p,$$
$$\|\Delta_\mu f(x,y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A$. Then there exists a unique derivation $D : A \to A$ such that

$$\|f(x) - D(x)\| \leq \frac{2^p\theta}{2-2^p}\|x\|^p$$

for all $x \in A$.

Proof. Set $\phi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in A$, and let $L = 2^{p-1}$ in Theorem **??**. Then we get the desired result. □

Remark 3.1. *Consider the function $f : \mathbb{C} \to \mathbb{C}$ which has been defined in Proposition* **??**. *Since $f(\bar{x}) = \overline{f(x)}$ for all $x \in \mathbb{C}$, Corollaries* **??** *and* **??** *are not true for $p = 1$.*

4. Superstability

In this section, we prove the super stability of of Jordan homomorphisms (Jordan $*$-homomorphisms), and Jordan derivations (Jordan $*$-derivations) on Banach algebras (C^*-algebras) for the generalized Jensen–type functional equation. We need the following lemma in our main results.

Lemma 4.1. *Let X and Y be linear normed spaces. Let $\theta \geq 0$, p and q be real numbers with $q > 0$ and $p + q \neq 1$. Suppose $f : X \to Y$ satisfies $f(0) = 0$ and*

$$\|\Delta_\mu f(x,y)\| \leq \theta\|x\|^p.\|y\|^q \tag{4.1}$$

for all $\mu \in \mathbb{T}$ and all $x, y \in X$(by letting $\|x\|^0 = 1$ and $x \neq 0$ for $p < 0$). Then f is linear.

Proof. Letting $y = 0$ in (**??**), we obtain that

$$f(rx) = rf(x), \quad f(\mu x) = \mu f(x) \tag{4.2}$$

for all $\mu \in \mathbb{T}$ and all $x \in X$. Hence, we have from (**??**) and (**??**) that

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \theta\|x\|^p.\|y\|^q \tag{4.3}$$

for all $x, y \in X$ with $x \neq 0$. Since $f(r^n x) = r^n f(x)$ for all $x \in X$ and all integers n, we get from (**??**) that

$$\|f(x+y) + f(x-y) - 2f(x)\| \leq \theta\Big(\frac{r^{p+q}}{r}\Big)^n\|x\|^p.\|y\|^q$$

for all integers n and all $x, y \in X$ with $x \neq 0$. Therefore

$$f(x+y)+f(x-y)=2f(x)$$

for all $x, y \in X$ with $x \neq 0$. Since f is odd, the last equation holds for all $x, y \in X$. Hence f is additive and we conclude that f is linear by Lemma **??**. $\square$

Now, we prove the superstability problem for Jordan homomorphisms and derivations on Banach algebras.

Corollary 4.2. *Let $p, s \in \mathbb{R}$ and $\theta, q \in (0, \infty)$ with $p+q \neq 1, s \neq 2$. Let A, B be Banach algebras. Suppose $f : A \to B$ $(f : A \to A)$ satisfies $f(0) = 0$ and*

$$\|\Delta_\mu f(x,y)\| \leq \theta \|x\|^p . \|y\|^q,$$
$$\|f(x^2)-(f(x))^2\| \leq \theta\|x\|^s, \quad \Big(\|f(x^2)-xf(x)-f(x)x\| \leq \theta \|x\|^s\Big)$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A$(by letting $\|x\|^0 = 1$ and $x \neq 0$ for $p < 0$ or $s < 0$). Then f is a Jordan homomorphism (Jordan derivation).

Proof. By Lemma **??**, f is linear. Hence,

$$\|f(x^2)-(f(x))^2\| \leq \theta n^{s-2}\|x\|^s, \quad \Big(\|f(x^2)-xf(x)-f(x)x\| \leq \theta n^{s-2} \|x\|^s\Big)$$

for all integers n and all $x \neq 0$. Therefore

$$f(x^2) = (f(x))^2, \quad \Big(f(x^2) = xf(x)+f(x)x\Big)$$

we get the desired result . $\square$

Corollary 4.3. *Let $p \in \mathbb{R}$ and $\theta, q \in (0, \infty)$ with $p+q \neq 1, 2$. Suppose A, B are C^*-algebras and $f : A \to B$ satisfies $f(0) = 0$ and*

$$\max\{\|f(x^*)-(f(x))^*\|, \|f(x^2)-(f(x))^2\|, \|\Delta_\mu f(x,y)\|\} \leq \theta\|x\|^p.\|y\|^q$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A$. Then f is a Jordan $*$-homomorphism.

Similarly, we can show that every nearly Jordan $*$-derivation is a derivation.

Corollary 4.4. *Let $p \in \mathbb{R}$ and $\theta, q \in (0, \infty)$ with $p+q \neq 1, 2$. Suppose A is a C^*-algebra and $f : A \to A$ satisfies $f(0) = 0$ and*

$$\max\{\|f(x^*)-(f(x))^*\|, \|f(x^2)-xf(x)-f(x)x\|, \|\Delta_\mu f(x,y)\|\} \leq \theta\|x\|^p.\|y\|^q$$

for all $\mu \in \mathbb{T}$ and all $x, y \in A$. Then f is a $*$-derivation.

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