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Fusion integral

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In this paper we shall introduce an operator-valued integral over a Hilbert space, called the fusion integral. We shall prove that it is additive (finitely or infinitely) but satisfies no additional condition. We shall show that it has the retrieval property and establish a useful connection between the fusion and the Lebesgue integrals.

1 Introduction

Throughout this paper H will be a Hilbert space and $\mathbb{H}$ will be the collection of all closed subspaces of H, respectively. Also, (X, μ) will be a measure space and, $v : X \to [0, \infty)$ a measurable mapping such that $v \neq 0$ a.e., We shall denote the closed unit ball of H by H_1.

We shall use the following lemmas, which can be found in [2] and [8].

Lemma 1.1 *Let K be a Hilbert space. Let $u : K \to H$ be a bounded operator with closed range $\mathcal{R}_u$. Then there exists a bounded operator $u^\dagger : H \to K$ for which*

$$uu^\dagger f = f, \quad f \in \mathcal{R}_u.$$

Also, $u^* : H \to K$ has closed range and $(u^*)^\dagger = (u^\dagger)^*$.

The operator $u^\dagger$ is called the pseudo-inverse of u.

Lemma 1.2 *Let $u : K \to H$ be a bounded surjective operator. Given $y \in H$, the equation $ux = y$ has a unique solution of minimal norm, namely, $x = u^\dagger y$.*

Lemma 1.3 *Let $u : K \to H$ be a bounded operator. Then*:

(i) $||u|| = ||u^*||$ *and* $||uu^*|| = ||u||^2$.

(ii) $\mathcal{R}_u$ *is closed, if and only if,* $\mathcal{R}_{u^*}$ *is closed.*

(iii) u *is surjective if and only if there exists* $c > 0$ *such that for each* $h \in H$

$$c\,||h|| \le ||u^*(h)||.$$

Lemma 1.4 *Let u be a self-adjoint bounded operator on H. Let*

$$m_u = \inf_{h \in H_1} \langle uh, h\rangle \quad \textit{and} \quad M_u = \sup_{h \in H_1} \langle uh, h\rangle.$$

Then, $m_u, M_u \in \sigma(u)$.

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Definition 1.5 Let $F : X \to \mathbb{H}$. Let $L^2(X, H, F)$ be the class of all measurable mappings $f : X \to H$ such that for each $x \in X$, $f(x) \in F(x)$and

$$\int_X \|f(x)\|^2 \, d\mu < \infty.$$

It can be verified that $L^2(X, H, F)$ is a Hilbert space with the scalar product defined by

$$\langle f, g \rangle = \int_X \langle f(x), g(x) \rangle \, d\mu, \quad f, g \in L^2(X, H, F).$$

Definition 1.6 Let $\{H_i\}_{i \in I}$ be a collection of Hilbert spaces. We denote their Hilbert space direct sum by $\bigoplus_i H_i$ and we define it as the collection of all mappings $f : I \to \bigcup_i H_i$ such that for each $i \in I$, $f_i \in H_i$ and $\sum_i ||f_i||^2 < \infty$, which is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_i \langle f_i, g_i \rangle, \quad f, g \in \bigoplus_i H_i.$$

We may say that the Hilbert space $L^2(X, H, F)$ is indeed, the Hilbert space continuous direct sum of $\{F(x)\}_{x \in X}$.

Definition 1.7 Let $\mathcal{L}(X, H, v)$ be the class of all mappings $F : X \to \mathbb{H}$ such that for each $h \in H$ the mapping $x \to \pi_{F(x)}(h)$ is measurable (i.e., F is weakly measurable) and

$$\sup_{h \in H_1} \int_X v^2(x) \|\pi_{F(x)}(h)\|^2 \, d\mu < \infty,$$

where, for each $x \in X$, $\pi_{F(x)}$ is the orthogonal projection onto $F(x)$.

Remark 1.8 For brevity, we shall denote $L^2(X, H, F)$ and $\mathcal{L}(X, H, v)$ by $L^2(X, F)$ and $\mathcal{L}(X, v)$, respectively.

Let $F \in \mathcal{L}(X, v)$, $f \in L^2(X, F)$ and $h \in H$. Then:

$$\begin{aligned}
\left| \int_X v(x) \langle f(x), h \rangle \, d\mu \right| &= \left| \int_X v(x) \big\langle \pi_{F(x)}(f(x)), h \big\rangle \, d\mu \right| \\
&= \left| \int_X v(x) \big\langle f(x), \pi_{F(x)}(h) \big\rangle \, d\mu \right| \\
&\leq \int_X v(x) ||f(x)||.||\pi_{F(x)}(h)|| \, d\mu \\
&\leq \left(\int_X ||f(x)||^2 \, d\mu \right)^{1/2} \left(\int_X v^2(x) \big\|\pi_{F(x))}\big\|^2 \, d\mu \right)^{1/2} \\
&\leq \left(\int_X ||f(x)||^2 \, d\mu \right)^{1/2} \sup_{h \in H_1} \left(\int_X v^2(x) \big\|\pi_{F(x)}(h)\big\|^2 \, d\mu \right)^{1/2} ||h||.
\end{aligned}$$

So we may define:

Definition 1.9 Let $F \in \mathcal{L}(X, v)$. We shall denote the fusion integral of F over X by $\oint_X F \, d\mu$ as the linear operator of $L^2(X, F)$ to H defined by

$$\left\langle \oint_X F \, d\mu(f), h \right\rangle = \int_X v(x) \langle f(x), h \rangle \, d\mu, \quad f \in L^2(X, F), \quad h \in H.$$

We shall denote the adjoint of $\oint_X F \, d\mu$ by $\oint_X^* F \, d\mu$.

Lemma 1.10 *Let $F \in \mathcal{L}(X, v)$. Then, $\oint_X F \, d\mu$ is bounded and $\oint_X^* F \, d\mu = v\pi_F$.*

P r o o f. Let $f \in L^2(X,F)$. We have

$$\begin{aligned}\left\|\oint_X F\,d\mu(f)\right\| &= \sup_{k\in H_1}\left|\left\langle \oint_X F\,d\mu(f),k\right\rangle\right| \\ &= \sup_{k\in H_1}\left|\int_X v(x)\langle f(x),k\rangle\,d\mu\right| \\ &= \sup_{k\in H_1}\left|\int_X v(x)\langle f(x),\pi_{F(x)}(k)\rangle\,d\mu\right| \\ &\le \sup_{k\in H_1}\left(\int_X v(x)^2\|\pi_{F(x)}(k)\|^2 d\mu\right)^{1/2}\left(\int_X \|f(x)\|^2 d\mu\right)^{1/2}.\end{aligned}$$

Thus,

$$\left\|\oint_X F\,d\mu\right\|^2 \le \sup_{k\in H_1}\int_X v(x)^2\|\pi_{F(x)}(k)\|^2\,d\mu < \infty.$$

Since $\oint_X F\,d\mu$ is bounded, for each $h \in H$ and $f \in L^2(X,F)$, we have

$$\begin{aligned}\left\langle \oint_X^* F\,d\mu(h),f\right\rangle &= \left\langle h,\oint_X F\,d\mu(f)\right\rangle = \int_X v(x)\langle h,f(x)\rangle\,d\mu \\ &= \int_X v(x)\langle \pi_{F(x)}(h),f(x)\rangle\,d\mu = \langle v\pi_F(h),f\rangle.\end{aligned}$$

Thus,

$$\oint_X^* F\,d\mu = v\pi_F.$$

□

Definition 1.11 Let $F,G \in \mathcal{L}(X,v)$. We say that F,G are *weakly equal* if

$$\oint_X^* F\,d\mu = \oint_X^* G\,d\mu,$$

which is equivalent to

$$v\pi_F(h) = v\pi_G(h), \quad \text{a.e.}$$

for all $h \in H$. Since, $v \neq 0$ a.e., F,G are weakly equal if

$$\pi_F(h) = \pi_G(h), \quad \text{a.e.}$$

for all $h \in H$.

Remark 1.12 Let $\int_X F\,d\mu = 0$. Now, let $O : X \to \mathbb{H}$ be defined by

$$O(x) = \{0\},$$

for all $x \in X$. Then $O \in \mathcal{L}(X,v)$ and $\oint_X O\,d\mu = 0$. Let $h \in H$. Since, $v\pi_F(h) \in L^2(X,F)$, so

$$\begin{aligned}\int_X v(x)^2\langle \pi_{F(x)}(h),\pi_{F(x)}(h)\rangle\,d\mu &= \int_X v(x)\langle v(x)\pi_{F(x)}(h),h\rangle\,d\mu \\ &= \left\langle \oint_X F\,d\mu(v\pi_F(h),h\right\rangle = 0.\end{aligned}$$

Thus, $\pi_{F(x)}(h) = 0$, a.e. Therefore,

$$\pi_F(h) = \pi_O(h), \quad \text{a.e.}$$

Hence, $F = O$, weakly.

2 Optimal lower and upper bounds

In this section, we shall show some interesting properties of the fusion integral. We shall prove an equivalence relation between the optimal lower and upper bounds of square norm of the conjugate of the fusion integral on the closed unit ball of H and optimal lower and upper bounds of Lebesgue integral of square modulus of a desired two variable mapping.

Definition 2.1 For each $F \in \mathcal{L}(X, v)$, we shall denote

$$\begin{aligned} A_{F,v} &= \inf\{\|v\pi_F(h)\|^2 : h \in H, ||h|| = 1\}, \\ B_{F,v} &= \sup\{\|v\pi_F(h)\|^2 : h \in H, ||h|| = 1\} = ||v\pi_F||^2. \end{aligned}$$

Remark 2.2 Let $F \in \mathcal{L}(X, v)$. Since, for each $h \in H$

$$\left\langle \oint_X F\, d\mu \oint_X^* F\, d\mu(h), h \right\rangle = ||v\pi_F(h)||^2,$$

$A_{F,v}$ and $B_{F,v}$ are optimal scalars which satisfy

$$A_{F,v} \leq \oint_X F\, d\mu \int_X^* F\, d\mu \leq B_{F,v}.$$

Lemma 2.3 *Let $F \in \mathcal{L}(X, v)$. Then $A_{F,v} > o$ if and only if $\oint_X F\, d\mu$ is surjective.*

P r o o f. Let $A_{F,v} > o$. Since, for each $h \in H$

$$\begin{aligned} \left\langle \oint_X F\, d\mu \oint_X^* F\, d\mu(h), h \right\rangle &= \int_X v^2(x)||\pi_{F(x)}(h)\|^2\, d\mu \\ &= \|v\pi_F(h)\|^2 \geq A_{F,v}\langle h, h\rangle. \end{aligned}$$

Therefore, by the Lemma 1.3(iii)

$$\oint_X F\, d\mu : L^2(X, F) \longrightarrow H$$

is surjective.

Now let $\oint_X F\, d\mu$ be surjective. Let

$$\oint_X^\dagger F\, d\mu : H \longrightarrow L^2(X, F)$$

be its pseudo-inverse. By the Lemma 1.1, for each $h \in H$

$$\begin{aligned} \|h\| &= \left\| \oint_X^{\dagger^*} F\, d\mu \oint_X^* F\, d\mu(h) \right\| \\ &\leq \left\| \oint_X^{\dagger^*} F\, d\mu \right\| \left\| \oint_X^* F\, d\mu(h) \right\| \\ &= \left\| \oint_X^{\dagger^*} F\, d\mu \right\| \|v\pi_F(h)\|. \end{aligned}$$

So

$$A_{F,v} \geq \left\| \int_X^{\dagger^*} F\, d\mu \right\|^{-2} > 0.$$

□

Theorem 2.4 *Let $F \in \mathcal{L}(X, H, v)$ and let K be a Hilbert space. Let $u : H \to K$ be a bounded bijective linear operator and $uF \in \mathcal{L}(X, K, v)$. Then*:

(i) $uL^2(X, H, F) = L^2(X, K, uF)$.

(ii) *For each $f \in L^2(X, F)$*

$$u \oint_X F\, d\mu(f) = \oint_X uF\, d\mu(uf).$$

(iii) $A_{F,v} > o$ *if and only if* $A_{uF,v} > o$.

Proof. (i) It is straightforward.

(ii) For each $k \in K$, we have

$$\begin{aligned}\left\langle u \oint_X F\, d\mu(f), k \right\rangle &= \left\langle \oint_X F\, d\mu(f), u^*(k) \right\rangle = \int_X v(x)\langle f(x), u^*(k)\rangle \\ &= \int_X v(x)\langle uf(x), k\rangle = \left\langle \oint_X uF\, d\mu(uf), k \right\rangle .\end{aligned}$$

Hence

$$u \oint_X F\, d\mu(f) = \oint_X uF\, d\mu(uf).$$

(iii) It is clear from (ii) and Lemma 2.3. □

Lemma 2.5 *Let $F \in \mathcal{L}(X, v)$. Then the operator $\oint_X F\, d\mu \oint_X^* F\, d\mu$ is invertible if and only if $A_{F,v} > o$.*

Proof. Let $\oint_X F\, d\mu \oint_X^* F\, d\mu$ be invertible. We have

$$\begin{aligned}A_{F,v} &= \inf_{h \in H_1} \left\| \oint_X^* F\, d\mu(h) \right\|^2 \\ &= \inf_{h \in H_1} \left\langle \oint_X F\, d\mu \oint_X^* F\, d\mu(h), h \right\rangle \in \sigma\left(\oint_X F\, d\mu \oint_X^* F\, d\mu \right),\end{aligned}$$

so, $A_{F,v} > 0$. Now let $A_{F,v} > o$. So, by the Lemma 2.3, $\oint_X F\, d\mu$ is surjective. Then there exists $A > 0$ such that

$$A||h|| \le \left\| \oint_X^* F\, d\mu(h) \right\|, \quad h \in H.$$

Hence

$$A_{F,v} \ge A^2 > 0.$$

□

Theorem 2.6 *Let $\{H_i\}_{i \in I}$ be a collection of Hilbert spaces and let $H = \bigoplus_i H_i$. Let $F \in \mathcal{L}(X, v)$ be such that for each $i \in I$ there exists at most one $x \in X$ such that $F(x) \subset H_i$. Let $A_{F,v} > o$, and each finite subset of X be measurable. Then, for each $h \in H$*

$$h = \sum_{x \in X} \pi_{F(x)}(h).$$

Proof. Let

$$K = \left\{ h \in H : h = \sum_{x \in X} \pi_{F(x)}(h) \right\}.$$

Let $\{f_n\}$ be a sequence of members of K which tends to $f \in H$. Given $\epsilon > 0$, we can find $N > 0$ such that $||f_N - f|| < \epsilon$. There exists a finite $Z \subseteq X$ such that for each finite $Z \subseteq Y \subseteq X$,

$$\left\| f_N - \sum_{x\in Y} \pi_{F(x)} f_N \right\| < \epsilon.$$

We have

$$\begin{aligned} &\left\| f - \sum_{x\in Y} \pi_{F(x)}(f) \right\| \\ &\quad \leq \|f - f_N\| + \left\| f_N - \sum_{x\in Y} \pi_{F(x)}(f_N) \right\| + \left\| \sum_{x\in Y} \pi_{F(x)}(f) - \sum_{x\in Y} \pi_{F(x)}(f_N) \right\|. \end{aligned}$$

But

$$\begin{aligned} A_{F,v} \left\| \sum_{x\in Y} \pi_{F(x)}(f) - \sum_{x\in Y} \pi_{F(x)}(f_N) \right\|^2 &= A_{F,v} \left\| \sum_{x\in Y} \pi_{F(x)}(f_N - f) \right\|^2 \\ &\leq \int_X v^2(t) \left\| \pi_{F(t)} \sum_{x\in Y} \pi_{F(x)}(f_N - f) \right\|^2 d\mu \\ &= \int_Y v^2(t) \left\| \pi_{F(t)} \sum_{x\in Y} \pi_{F(x)}(f_N - f) \right\|^2 d\mu \\ &= \int_Y v^2(t) \left\| \pi_{F(t)}(f_N - f) \right\|^2 d\mu \leq B_{F,v} ||f_N - f||^2. \end{aligned}$$

So, K is a closed subspace of H. Now, let $h \in K^\perp$. Since, for each $t \in X$

$$\pi_{F(t)}(h) = \sum_{x\in X} \pi_{F(x)} \pi_{F(t)}(h),$$

$\pi_{F(t)}(h) \in K$. Since

$$\left\| \pi_{F(t)}(h) \right\|^2 = \left\langle h, \pi_{F(t)}(h) \right\rangle = 0$$

and $A_{F,v} > o$, $h = 0$. □

Theorem 2.7 *Let (X, μ) and (Y, λ) be two σ-finite measure spaces and let $f : X \times Y \to H, F : X \to \mathbb{H}$ be weakly measurable mappings. Suppose that for each $x \in X$ the function $f(x, .) : Y \to F(x)$ is measurable, that*

$$\begin{aligned} 0 < A(x) &= \inf_{h\in F(x)_1} \int_Y |\langle f(x,y), h\rangle|^2 \, d\lambda \\ &\leq \sup_{h\in F(x)_1} \int_Y |\langle f(x,y), h\rangle|^2 \, d\lambda = B(x) < \infty, \end{aligned}$$

and that

$$0 < A = \inf_x A(x) \leq \sup_x B(x) = B < \infty.$$

Then, $F \in \mathcal{L}(X, v)$ and $A_{F,v} > 0$ if and only if

$$\begin{aligned} o &< \inf_{h\in H_1} \int_{X\times Y} |\langle v(x) f(x,y), h\rangle|^2 \, d(\mu \times \lambda) \\ &\leq \sup_{h\in H_1} \int_{X\times Y} |\langle v(x) f(x,y), h\rangle|^2 \, d(\mu \times \lambda) < \infty. \end{aligned}$$

P r o o f. Since the spaces (X,μ) and (Y,λ) are σ-finite, by the Fubini's theorem we have

$$\begin{aligned}
A\|v\pi_F(h)\|^2 &= A\int_X v^2(x)\big\|\pi_{F(x)}(h)\big\|^2 d\mu\\
&\le \int_X A(x)v^2(x)\big\|\pi_{F(x)}(h)\big\|^2 d\mu\\
&\le \int_X\int_Y \big|\langle v(x)\pi_{F(x)}(h), f(x,y)\rangle\big|^2 d\lambda\, d\mu\\
&= \int_X\int_Y |\langle h, v(x)f(x,y)\rangle|^2 d\lambda\, d\mu\\
&= \int_{X\times Y} |\langle v(x)f(x,y), h\rangle|^2 d(\mu\times\lambda)\\
&= \int_X\int_Y \big|\langle \pi_{F(x)}(h), v(x)f(x,y)\rangle\big|^2 d\lambda\, d\mu\\
&\le \int_X B(x)v^2(x)\big\|\pi_{F(x)}(h)\big\|^2 d\mu\\
&\le B\int_X v^2(x)\big\|\pi_{F(x)}(h)\big\|^2 d\mu,
\end{aligned}$$

and the theorem is proved. □

Theorem 2.8 *Let (X,μ) be a σ-finite measure space. Let K be a Hilbert space. Let $u: H\to K$ be a bijective bounded linear operator. Let $F: X\to\mathbb{H}$ and $uF: X\to\mathbb{K}$ be weakly measurable. Then, $F\in\mathcal{L}(X,H,v)$ and $A_{F,v}>0$ if and only if $uF\in\mathcal{L}(X,K,v)$ and $A_{uF,v}>0$.*

P r o o f. Let $F\in\mathcal{L}(X,H,v)$ and let $A_{F,v}>0$. Let (Y,λ) be a σ-finite measure space and let $f: X\times Y\to H$ be such that for each $x\in X$, $f(x,.) : Y\to F(x)$ with

$$\begin{aligned}
0 < A(x) &= \inf_{h\in F(x)_1}\int_Y |\langle f(x,y),h\rangle|^2\, d\lambda\\
&\le \sup_{h\in F(x)_1}\int_Y |\langle f(x,y),h\rangle|^2\, d\lambda = B(x) < \infty
\end{aligned}$$

measurable and

$$0 < A = \inf_x A(x) \le \sup_x B(x) = B < \infty.$$

Choosing such a mapping is always possible. Indeed, let $\{e_i^x\}_{i\in I_x}$ be an orthonormal basis for $F(x)$. We can suppose that $\{I_x\}_{x\in X}$ is pairwise disjoint (we can consider $\{x\}\times I_x$). Let $Y=\bigcup_{x\in X} I_x$, and λ be the counting measure in Y. Then we can define $f: X\times Y\to H$ by

$$f(x,i)=\begin{cases} e_i^x, & \text{if } i\in I_x,\\ 0, & \text{otherwise.}\end{cases}$$

Then, for each $x\in X$, $A(x)=B(x)=1$. By the Theorem 2.7

$$\begin{aligned}
o &< \inf_{h\in H_1}\int_{X\times Y} |\langle v(x)f(x,y),h\rangle|^2\, d(\mu\times\lambda)\\
&\le \sup_{h\in H_1}\int_{X\times Y} |\langle v(x)f(x,y),h\rangle|^2\, d(\mu\times\lambda) < \infty.
\end{aligned}$$

Then, $uf : X \times Y \to \mathbb{K}$, and for each $x \in X, uf(x,.) : Y \to uF(x)$. Since u is surjective, there is $C > 0$ such that

$$\begin{aligned} & C^2 ||h||^2 \int_Y |\langle f(x,y), u^*(h)/||u^*(h)||\rangle|^2 \, d\lambda \\ & \quad \le \int_Y |\langle f(x,y), u^*(h)/||u^*(h)||\rangle|^2 \, d\lambda \, ||u^*(h)||^2 \\ & \quad = \int_Y |\langle uf(x,y), h\rangle|^2 \, d\lambda \le ||u||^2 ||h||^2 \int_Y |\langle f(x,y), u^*(h)/||u^*(h)||\rangle|^2 d\lambda. \end{aligned}$$

So,

$$\begin{aligned} C^2 A(x) \; & \le \; \inf_{h\in H_1} \int_Y |\langle uf(x,y), h\rangle|^2 \, d\lambda \\ & \le \; \sup_{h\in H_1} \int_Y |\langle uf(x,y), h\rangle|^2 \, d\lambda \le ||u^2|| B(x). \end{aligned}$$

Similarly, we have

$$\begin{aligned} & C^2 \inf_{h\in H_1} \int_{X\times Y} |\langle v(x) f(x,y), h\rangle|^2 \, d(\mu\times\lambda) \\ & \quad \le \inf_{h\in H_1} \int_{X\times Y} |\langle v(x) uf(x,y), h\rangle|^2 \, d(\mu\times\lambda) \\ & \quad \le \sup_{h\in H_1} \int_{X\times Y} |\langle v(x) uf(x,y), h\rangle|^2 \, d(\mu\times\lambda) \\ & \quad \le ||u||^2 \sup_{h\in H_1} \int_{X\times Y} |\langle v(x) uf(x,y), h\rangle|^2 \, d(\mu\times\lambda). \end{aligned}$$

Therefore, by the Theorem 2.7 $uF \in \mathcal{L}(X,K,v)$ and $A_{uF,v} > 0$. The proof of the converse is similar. □

3 Uncountable additivity

In this section, we shall show that the fusion integral has retrieval property. Also, we shall show that it is uncountably additive.

Theorem 3.1 *Let $F \in \mathcal{L}(X,v)$ and let $A_{F,v} > 0$. Let $h \in H$ and let $u = \oint_X F \, d\mu \oint_X^* F \, d\mu$. Then:*

(i) *We have the following retrieval formulas*

$$h = \oint_X u^{-1} F \, d\mu(u^{-1} v\pi_F(h))$$

and

$$h = \oint_X F \, d\mu(v\pi_F(u^{-1}(h)).$$

(ii) *In the retrieval formula $h = \oint_X F \, d\mu(v\pi_F(u^{-1}(h)), v\pi_F(u^{-1}(h))$ has least norm among all of the retrieval formulas.*

(iii) *For each $h \in H$, $\oint_X^\dagger F \, d\mu(h) = v\pi_F(u^{-1}(h))$.*

Proof. (i) Since $A_{F,v} > 0$, u is an invertible operator. We have

$$h = u^{-1}u(h) = u^{-1}\oint_X F\,d\mu(v\pi_F(h)) = \oint_X u^{-1}F\,d\mu(u^{-1}v\pi_F(h)).$$

Also, we have

$$h = uu^{-1}(h) = \oint_X F\,d\mu(v\pi_F(u^{-1}(h)).$$

(ii) Let $f \in L^2(X,F)$ and let $h = \oint_X F\,d\mu(f)$. Thus, for each $k \in H$ we have

$$\begin{aligned}\langle h,k\rangle &= \left\langle \oint_X F\,d\mu(f),k\right\rangle \\ &= \int_X v(x)\langle f(x),k\rangle\,d\mu \\ &= \left\langle \oint_X F\,d\mu(v\pi_F(u^{-1}(h)),k\right\rangle \\ &= \int_X v(x)\big\langle v(x)\pi_{F(x)}(u^{-1}(h)),k\big\rangle\,d\mu.\end{aligned}$$

Therefore

$$\left\langle \oint_X F\,d\mu(v\pi_F(u^{-1}(h))-f),k\right\rangle = 0.$$

So

$$\oint_X F\,d\mu(v\pi_F(u^{-1}(h))-f) = 0.$$

Hence $v\pi_F(u^{-1}(h)) - f \in ker \oint_X F\,d\mu$. Since, $F \in \mathcal{L}(X,v)$,

$$v\pi_F(u^{-1}(h)) \in \mathcal{R}\oint_X F\,d\mu.$$

But,

$$L^2(X,F) = \Big(ker\oint_X F\,d\mu\Big) \oplus \Big(\mathcal{R}\oint_X^{*} F\,d\mu\Big).$$

So

$$||f||^2 = ||v\pi_F(u^{-1}(h)) - f||^2 + ||v\pi_F(u^{-1}(h))||^2,$$

and (ii) is proved.

(iii) Let $f \in L^2(X,F)$. Since, $\oint_X^{\dagger} F\,d\mu(h)$ is the unique solution of minimal norm of

$$\oint_X F\,d\mu(f) = h,$$

so by (ii)

$$\int_X |f - v\pi_F(u^{-1}(h))|^2 d\mu = 0.$$

Therefore,

$$f = v\pi_F(u^{-1}(h)) = \oint_X^{\dagger} F\,d\mu(h).$$

□

Theorem 3.2 *Let $F, G \in \mathcal{L}(X, v)$. Then the following assertions are equivalent*:

(i) *For each $h \in H$, $h = \oint_X F\, d\mu(v\pi_F\pi_G(h))$.*

(ii) *For each $h \in H$, $h = \oint_X G\, d\mu(v\pi_G\pi_F(h))$.*

(iii) *For each $h, k \in H$, $\langle h, k\rangle = \int_X v^2\langle \pi_G(h), \pi_F(k)\rangle\, d\mu$.*

(iv) *For each $h \in H$, $||h||^2 = \int_X v^2\langle \pi_G(h), \pi_F(h)\rangle\, d\mu$.*

(v) *For each orthonormal bases $\{e_i\}_{i\in I}$ and $\{\gamma_j\}_{j\in J}$ for H*

$$\langle e_i, \gamma_j\rangle = \int_X v^2\langle \pi_F(e_i), \pi_G(\gamma_j)\rangle\, d\mu, \quad i \in I, \quad j \in J.$$

(vi) *For each orthonormal basis $\{e_i\}_{i\in I}$ for H*

$$\int_X v^2\langle \pi_F(e_i), \pi_G(e_i)\rangle\, d\mu = 1, \quad i \in I.$$

Proof. (i) ⇒ (ii) Let $h, k \in H$. We have

$$\begin{aligned}\langle h, k\rangle &= \left\langle \oint_X F\, d\mu(v\pi_F\pi_G(h)), k\right\rangle = \int_X v\langle v\pi_F\pi_G(h), k\rangle\, d\mu \\ &= \int_X v\langle h, v\pi_G\pi_F(k)\rangle\, d\mu = \left\langle h, \oint_X G\, d\mu(v\pi_G\pi_F(k))\right\rangle.\end{aligned}$$

Hence, $k = \oint_X G\, d\mu(v\pi_G\pi_F(k))$.

(ii) ⇒ (iii) It is evident by the proof of (i)⇒ (ii).

(iii) ⇒ (i) For each $h, k \in H$, we have

$$\begin{aligned}\langle h, k\rangle &= \int_X v^2\langle \pi_G(h), \pi_F(k)\rangle\, d\mu \\ &= \left\langle \oint_X F\, d\mu(v\pi_F\pi_G(h)), k\right\rangle.\end{aligned}$$

Thus $h = \oint_X F\, d\mu(v\pi_F\pi_G(h)$.

(iv) ⇒ (i) Let $L : H \to H$ be defined by

$$L(h) = \oint_X F\, d\mu(v\pi_F\pi_G(h)).$$

It is clear that L is linear. Since

$$\begin{aligned}||L(h)|| &= \sup_{k\in H_1} |\langle L(h), k\rangle| = \sup_{k\in H_1} \left|\int_X v^2\langle \pi_F\pi_G(h), k\rangle\, d\mu\right| \\ &\le \left(\int_X v^2||\pi_G(h)||^2 d\mu\right)^{1/2} \sup_{k\in H_1} \left(\int_X v^2||\pi_F(k)||^2 d\mu\right)^{1/2} \\ &\le \sup_{h\in H_1} \left(\int_X v^2||\pi_G(h)||^2 d\mu\right)^{1/2} \sup_{k\in H_1} \left(\int_X v^2||\pi_F(k)||^2 d\mu\right)^{1/2} ||h|| \\ &= B_{F,v}^{1/2} B_{G,v}^{1/2} ||h||,\end{aligned}$$

so that, $L \in B(H)$. For each $h \in H$, we have

$$\langle h, h\rangle = ||h||^2 = \int_X v^2\langle \pi_G(h), \pi_F(h)\rangle\, d\mu = \left\langle \oint_X f\, d\mu(v\pi_F\pi_G(h), h\right\rangle.$$

Hence, for each $h \in H, h = \oint_X F\, d\mu(v\pi_F\pi_G(h))$.

(iii) $\Rightarrow$ (iv) is evident.

(v) $\Rightarrow$ (iii) We have

$$\begin{aligned}\int_X v^2\langle \pi_F(h), \pi_G(k)\rangle\, d\mu &= \langle v\pi_F(h), v\pi_G(k)\rangle\\ &= \left\langle v\pi_F\left(\sum_i \langle h, e_i\rangle e_i\right), v\pi_G\left(\sum_j \langle k, \gamma_j\rangle \gamma_j\right)\right\rangle\\ &= \sum_{i,j}\langle\langle h, e_i\rangle v\pi_F(e_i), \langle k, \gamma_j\rangle v\pi_G(\gamma_j)\rangle\\ &= \sum_{i,j}\langle h, e_i\rangle\langle \gamma_j, k\rangle\langle v\pi_F(e_i), v\pi_G(\gamma_j)\rangle\\ &= \sum_{i,j}\langle h, e_i\rangle\langle \gamma_j, k\rangle\langle e_i, \gamma_j\rangle\\ &= \langle h, k\rangle.\end{aligned}$$

(vi) $\Rightarrow$ (v) It is similar with the proof of (v) $\Rightarrow$ (iii). □

Definition 3.3 Let $F, G \in \mathcal{L}(X, v)$. We say that F, G is a *fusion pair* if one of the assertions of Theorem 3.2 satisfied.

Lemma 3.4 *Let F, G be a fusion pair. Then $A_{F,v} > 0$.*

P r o o f. By Theorem 3.2(iv), for each $h \in H$, we have

$$||h||^2 = \int_X \langle v\pi_F(h), v\pi_G(h)\rangle\, d\mu \le B_{G,v}^{1/2}(||v\pi_F(h)||)||h||.$$

Since, $G \in \mathcal{L}(X, v)$,

$$A_{F,v} = \inf_{h\in H_1} ||v\pi_F(h)||^2 \ge B_{G,v}^{-1} > 0.$$

□

Definition 3.5 Let $\{H_i\}_{i\in I}$ be a collection of Hilbert spaces, let $H = \bigoplus H_i$ and let $J \subseteq I$. Let $\{F_j\}_{j\in J}$ be a collection of mappings $F_j : X \to \mathbb{H}$. Then, we denote the direct sum of $\{F_j\}_j$ by $\bigoplus_j F_j$ and we define it by

$$\bigoplus_j F_j : X \longrightarrow \mathbb{H}, \quad \Big(\bigoplus_j F_j\Big)(x) = \bigoplus_j F_j(x).$$

Definition 3.6 Let $\{H_i\}_{i\in I}$ be a collection of Hilbert spaces, let $H = \bigoplus H_i$ and let $J \subseteq I$. Let $\{F_j\}_{j\in J}$ be a collection of mappings $F_j : X \to \mathbb{H}$. We denote the function space direct sum of $\{L^2(X, H, F_j)\}_j$ by $\hat{\bigoplus}_j L^2(X, H, F_j)$ and we define the class of all measurable mappings $\bigoplus_j f_j : X \to H$ such that for each $x \in X, \bigoplus_j f_j(x) = \{f_j(x)\}_j$, $f_j \in L^2(X, H, F_j)$ and $\sum_j ||f_j||^2 < \infty$.

Theorem 3.7 *Let $\{H_i\}_{i\in I}$ be a collection of Hilbert spaces, let $H = \bigoplus_i H_i$ and let $J \subseteq I$. Let $\{F_j\}_{j\in J}$ be a collection of mappings $F_j : X \to \mathbb{H}$ such that for each $j \in J$ and for each $x \in X, F_j(x) \in \{H_i\}$. Then*:

(i) $L^2\big(X, H, \bigoplus_j F_j\big) = \hat{\bigoplus}_j L^2(X, H, F_j)$.

(ii) *Let $\bigoplus_j F_j \in \mathcal{L}(X, v)$. Then, $\oint_X \bigoplus_j F_j\, d\mu = \bigoplus_j \oint_X F_j\, d\mu$.*

P r o o f. (i) Let $f \in L^2\big(X, H, \bigoplus_j F_j\big)$. Then, $f : X \to H$ is measurable and for each $x \in X$, $f(x) \in \big(\bigoplus_j F_j\big)(x) = \bigoplus_j (F_j(x))$ and

$$\int_X ||f(x)||^2\, d\mu = ||f||^2 < \infty.$$

Hence, $f(x) = \{f_j(x)\}_j$, where $f_j = \pi_{F_j} f : X \to H$ is measurable, $f_j(x) \in F_j(x)$ and

$$\sum_j ||f_j(x)||^2 = ||f(x)||^2 < \infty, \quad x \in X.$$

Since

$$\begin{aligned}\sum_j ||f_j||^2 &= \sum_j \int_X ||f_j(x)||^2 \, d\mu = \int_X \sum_j ||f_j(x)||^2 d\mu \\ &= \int_X ||f(x)||^2 \, d\mu = ||f||^2 < \infty,\end{aligned}$$

for each $j \in J$, $f_j \in L^2(X, H, F_j)$ and $\sum_j ||f_j||^2 < \infty$. Hence

$$f = \bigoplus_j f_j \in \hat{\bigoplus_j} L^2(X, H, F_j).$$

Now, let $f = \bigoplus_j f_j \in \hat{\bigoplus}_j L^2(X, H, F_j)$. We have

$$f(x) \in \bigoplus_j (F_j(x)) = \Big(\bigoplus_j F_j\Big)(x), \quad x \in X,$$

and

$$\int_X ||f(x)||^2 \, d\mu = \int_X \sum_j ||f_j(x)||^2 \, d\mu = \sum_j \int_X ||f_j(x)||^2 \, d\mu = \sum_j ||f_j||^2 < \infty.$$

Hence, $f \in L^2\big(X, H, \bigoplus_j F_j\big)$.

(ii) Let $f \in L^2\big(X, H, \bigoplus_j F_j\big)$. Then, by (i), $f = \bigoplus_j f_j, f_j \in L^2(X, H, F_j)$ and $\sum_j ||f_j||^2 < \infty$. Let $h = \{h_i\}_i \in H$. We have

$$\begin{aligned}\Big\langle \oint_X \bigoplus_j F_j \, d\mu(f), h \Big\rangle &= \int_X v(x)\langle f(x), h\rangle \, d\mu \\ &= \int_X v(x)\langle \{f_j(x)\}_j, h\rangle \, d\mu = \int_X v(x)\langle \{f_i(x)\}_i, \{h_i\}_i\rangle \, d\mu \\ &= \int_X \sum_i v(x)\langle f_i(x), h_i\rangle \, d\mu = \int_X \sum_j v(x)\langle f_j(x), h_j\rangle \, d\mu,\end{aligned}$$

where, for each $i \in I - J$, we define $f_i = 0$ Since,

$$\begin{aligned}\sum_j \int_X v(x)|\langle f_j(x), h_j\rangle| \, d\mu &\le \sum_j \left[\left(\int_X ||f_j(x)||^2 \, d\mu\right)^{1/2} \left(\int_X v^2(x)\big\|\pi_{F_j(x)}(h_j)\big\|^2 \, d\mu\right)^{1/2}\right] \\ &= \sum_j \left[||f_j|| \left(\int_X v^2(x)\big\|\pi_{F_j(x)}(h_j)\big\|^2 \, d\mu\right)^{1/2}\right] \\ &\le \left(\sum_j ||f_j||^2\right)^{1/2} \left(\sum_j \int_X v^2(x)\big\|\pi_{F_j(x)}(h_j)\big\|^2 \, d\mu\right)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\sum_j \int_X v^2(x)\big\langle \pi_{F_j(x)}(h_j), \pi_{F_j(x)}(h_j)\big\rangle \, d\mu &= \int_X v^2(x) \sum_j \big\langle \pi_{F_j(x)}(h_j), \pi_{F_j(x)}(h_j)\big\rangle \, d\mu \\ &= \int_X v^2(x)\big\langle \big\{\pi_{F_j(x)}(h_j)\big\}_j, \{h_j\}_j\big\rangle \, d\mu \\ &= \int_X v^2(x)\big\langle \pi_{\oplus_j F_j(x)}(h), h\big\rangle \, d\mu < \infty,\end{aligned}$$

thus

$$\begin{aligned}\left\langle \oint_X \oplus_j F_j \, d\mu(f), h \right\rangle &= \sum_j \int_X v(x)\langle f_j(x), h_j\rangle \, d\mu \\ &= \sum_i \int_X v(x)\langle f_i(x), h_i\rangle \, d\mu \\ &= \sum_i \left\langle \oint_X F_i \, d\mu(f_i), h_i \right\rangle \\ &= \left\langle \left\{ \oint_X F_i \, d\mu(f_i) \right\}_i, \{h_i\}_i \right\rangle \\ &= \left\langle \bigoplus_i \oint_X F_i \, d\mu\left(\bigoplus_i f_i \right), h \right\rangle \\ &= \left\langle \bigoplus_j \oint_X F_j \, d\mu\left(\bigoplus_j f_j \right), h \right\rangle.\end{aligned}$$

So, for each $f \in L^2\big(X, H, \bigoplus_j F_j\big)$,

$$\oint_X \bigoplus_j F_j \, d\mu(f) = \bigoplus_j \oint_X f_j \, d\mu\big(\bigoplus f_j\big),$$

and the theorem is proved. □

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