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g-Riesz dual sequences for g-Bessel sequences

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Duality principles in Gabor theory play a fundamental role for analyzing Gabor systems. In this paper, we obtain some results of duality principles for g-frames. Let $\{\gamma_j \in B(H, H_j) : j \in \mathbb{N}\}$ and $\{\eta_i \in B(H, H_i) : i \in \mathbb{N}\}$ be g-orthonormal bases for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. For a g-Bessel sequence $\Lambda = \{\Lambda_i \in B(H, H_i) : i \in \mathbb{N}\}$, we define $\Theta_j^\Lambda = \sum_{i=1}^{\infty} \gamma_j \Lambda_i^* \eta_i : H \to H_j$. Then we describe some properties of the sequence $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}}$. We also characterize the sequence $\{\Theta_j^\Psi\}_{j\in\mathbb{N}}$ when $\Psi = \{\Psi_i \in B(H, H_i) : i \in \mathbb{N}\}$ is a dual g-frame of $\{\Lambda_i\}_{i\in\mathbb{N}}$.

Keywords: g-Orthonormal basis; g-Riesz basis; g-Riesz dual sequence.

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1. Introduction

The origins of the frame theory date back to the work of Duffin and Schaeffer [3], where they are used in nonharmonic Fourier analysis. Recent interest in frames has been motivated in part by their utility in analyzing wavelet expansions [4]. Duality principles such as the Wexler–Raz biorthogonality relations [10] play a fundamental role for analyzing Gabor systems. Casazza, Kutyniok and Lammers [1] presented a general approach to derive duality principles in frame theory. Zhu, Xiao and Wang generalized their results to frames in Banach spaces. In this study, we extend some results of duality principles in [1] to g-frames and the results are similar to the duality principles in [1].

Throughout this paper, H and K are complex separable Hilbert spaces and $\{H_i\}_{i\in I}$ is a sequence of closed subspaces of K. I and J_i, for each $i \in I$, are subsets of $\mathbb{Z}$ and $B(H, H_i)$ is the collection of all bounded linear operators of H into H_i.

This paper is organized as follows: In Sec. 2, we recall some definitions and properties of g-frames which will be used in this paper. In Sec. 3, for every g-Bessel sequence, we define a corresponding sequence dependent on two g-orthonormal bases and characterize some properties of the first sequence in terms of the associated one. Finally, in Sec. 4, first we find a clear structure for dual g-frames of $\{\Lambda_i\}_{i\in\mathbb{N}}$ and then characterize g-Riesz dual sequences for theses duals with respect to two g-orthonormal bases.

2. Preliminaries

Definition 2.1 ([8]). We call a sequence $\{\Lambda_i \in B(H, H_i) : i \in I\}$ a generalized frame, or simply a g-frame, for H with respect to $\{H_i\}_{i\in I}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad f \in H. \tag{2.1}$$

The constants A and B are called the lower and upper g-frame bounds, respectively. The sequence $\{\Lambda_i\}_{i\in I}$ is called a g-Bessel sequence with bound B, if the second inequality in (2.1) is satisfied.

We say that $\{\Lambda_i\}_{i\in I}$ is a g-frame sequence, if it is a g-frame for $\overline{\text{span}}_{i\in I} \times \{\Lambda_i^*(H_i)\}$.

For each sequence $\{H_i\}_{i\in I}$, we define the space $(\sum_{i\in I} \oplus H_i)_{l_2}$ by

$$\left(\sum_{i\in I} \oplus H_i\right)_{l_2} = \left\{\{f_i\}_{i\in I} : f_i \in H_i,\ \ i \in I \text{ and } \sum_{i\in I} \|f_i\|^2 < \infty\right\},$$

with the inner product defined by

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle.$$

It is clear that $(\sum_{i\in I} \oplus H_i)_{l_2}$ is a Hilbert space.

We define the synthesis operator for a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ as follows:

$$T_\Lambda : \left(\sum_{i\in I} \oplus H_i\right)_{l_2} \to H, \quad T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* f_i,$$

the series converges unconditionally in the norm of H. It is easy to show that the adjoint operator of T_Λ is as follows:

$$T_\Lambda^* : H \to \left(\sum_{i\in I} \oplus H_i\right)_{l_2}, \quad T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in I},$$

T_Λ^* is called the analysis operator for $\Lambda = \{\Lambda_i\}_{i\in I}$. In [8], the g-frame operator S_Λ for a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ is defined as follows:

$$S_\Lambda : H \to H, \quad S_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f,$$

hence, we have $S_\Lambda = T_\Lambda T_\Lambda^*$.

If $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for H with respect to $\{H_i\}_{i\in I}$ with bounds A and B, then the g-frame operator $S_\Lambda : H \to H$ is bounded, self-adjoint and invertible. The canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$ is defined by $\tilde{\Lambda} = \{\tilde{\Lambda}_i\}_{i\in I}$, where $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, which is also a g-frame for H with respect to $\{H_i\}_{i\in I}$ with frame bounds B^{-1} and A^{-1}. Also, we have

$$f = \sum_{i\in I} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i\in I} \tilde{\Lambda}_i^* \Lambda_i f, \quad f \in H.$$

Definition 2.2 ([7]). Let $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ be g-Bessel sequences for H with respect to $\{H_i\}_{i\in I}$. Then $\{\Theta_i\}_{i\in I}$ is called a dual g-frame of $\{\Lambda_i\}_{i\in I}$, if

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f, \quad f \in H.$$

In this case, $\{\Lambda_i\}_{i\in I}$ and $\{\Theta_i\}_{i\in I}$ are also g-frames for H with respect to $\{H_i\}_{i\in I}$.

Definition 2.3 ([8]). We say that $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is a g-orthonormal basis for H with respect to $\{H_i\}_{i\in I}$, if it satisfies the following assertions:

$$\langle \Lambda_{i_1}^* f_{i_1}, \Lambda_{i_2}^* g_{i_2}\rangle = \delta_{i_1,i_2}\langle f_{i_1}, g_{i_2}\rangle, \quad f_{i_1} \in H_{i_1}, g_{i_2} \in H_{i_2}, i_1, i_2 \in I, \tag{2.2}$$

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2, \quad f \in H. \tag{2.3}$$

Remark 2.4. We note that if $\{\Lambda_i\}_{i\in I}$ is a g-orthonormal basis, then by (2.3),

$$\langle f, f\rangle = \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle = \sum_{i\in I}\langle \Lambda_i^* \Lambda_i f, f\rangle = \left\langle \sum_{i\in I} \Lambda_i^* \Lambda_i f, f \right\rangle, \quad f \in H.$$

So, $f = \sum_{i\in I} \Lambda_i^* \Lambda_i f$.

Definition 2.5 ([8]). We say that $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is g-complete, if

$$\{f : \Lambda_i f = 0,\ i \in I\} = \{0\}.$$

Definition 2.6 ([8]). We say that $\{\Lambda_i \in B(H, H_i) : i \in I\}$ is a g-Riesz basis for H with respect to $\{H_i\}_{i\in I}$, if it is g-complete and there exist constants $0 < A \leq B < \infty$, such that for each finite subset $J \subseteq I$ and $g_i \in H_i,\ i \in J$,

$$A \sum_{i\in J} \|g_i\|^2 \leq \left\| \sum_{i\in J} \Lambda_i^* g_i \right\|^2 \leq B \sum_{i\in J} \|g_i\|^2.$$

The constants A and B are called the g-Riesz basis bounds.

We call $\{\Lambda_i\}_{i\in I}$ a g-Riesz sequence, if it is a g-Riesz basis for $\overline{\text{span}}_{i\in I}\{\Lambda_i^*(H_i)\}$.

Theorem 2.7 ([2]). *Let $T_n : X \to Y$, $n \in \mathbb{N}$, be a sequence of bounded operators, which converges pointwise to a mapping $T : X \to Y$. Then T is linear and bounded. Furthermore, the sequence of norms $\|T_n\|$ is bounded and $\|T\| \leq \liminf_n \|T_n\|$.*

3. g-Riesz Dual Sequences

Let $\{\gamma_j \in B(H, H_j) : j \in \mathbb{N}\}$ and $\{\eta_i \in B(H, H_i) : i \in \mathbb{N}\}$ be g-orthonormal bases for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. Suppose that $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. For each $j \in \mathbb{N}$, we define $K_j^\Lambda : H_j \to H$ as follows:

$$K_j^\Lambda(h) = \sum_{i=1}^{\infty} \eta_i^* \Lambda_i \gamma_j^* h, \quad h \in H_j.$$

To show that K_j^Λ is well defined, we prove that $\{\sum_{i=1}^{n} \eta_i^* \Lambda_i \gamma_j^* h\}_{n=1}^{\infty}$ is a Cauchy sequence in H. For this, consider $m, n \in \mathbb{N}$, $n > m$. Since $\{\eta_i\}_{i\in\mathbb{N}}$ is a g-orthonormal basis, we have

$$\begin{aligned} \left\| \sum_{i=1}^{n} \eta_i^* \Lambda_i \gamma_j^* h - \sum_{i=1}^{m} \eta_i^* \Lambda_i \gamma_j^* h \right\|^2 &= \left\| \sum_{i=m+1}^{n} \eta_i^* \Lambda_i \gamma_j^* h \right\|^2 \\ &= \sum_{i=m+1}^{n} \|\Lambda_i \gamma_j^* h\|^2. \end{aligned}$$

Therefore, for all $h \in H_j$ and $j \in \mathbb{N}$, $\sum_{i=1}^{\infty} \eta_i^* \Lambda_i \gamma_j^* h$ is convergent in H and K_j^Λ is well defined. By Theorem 2.7, K_j^Λ is linear and bounded. Since $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-Bessel sequence and $\{\eta_i\}_{i\in\mathbb{N}}$ is a g-orthonormal basis, for all $j \in \mathbb{N}$ and $f \in H$, the series $\sum_{i=1}^{\infty} \gamma_j \Lambda_i^* \eta_i f$ is convergent in H_j and we have

$$(K_j^\Lambda)^*(f) = \sum_{i=1}^{\infty} \gamma_j \Lambda_i^* \eta_i f, \quad f \in H.$$

Definition 3.1. Let $\{\gamma_j\}_{j\in\mathbb{N}}$ and $\{\eta_i\}_{i\in\mathbb{N}}$ be g-orthonormal bases for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. Suppose that $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. We call $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}} = \{(K_j^\Lambda)^*\}_{j\in\mathbb{N}}$ the g-Riesz dual sequence for $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ with respect to $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\eta_i\}_{i\in\mathbb{N}})$.

Example 3.2. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for H and $\{f_j\}_{j\in\mathbb{N}}$ be a Bessel sequence for H. For each $j \in \mathbb{N}$, we define

$$\gamma_j f = \eta_j f = \langle f, e_j\rangle, \quad f \in H,$$

and

$$\Lambda_j f = \langle f, f_j\rangle, \quad f \in H.$$

Let $\Theta_j^\Lambda f = \sum_{i=1}^\infty \gamma_j \Lambda_i^* \gamma_i f = \sum_{i=1}^\infty \langle f, e_i\rangle\langle f_i, e_j\rangle$. Then $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}}$ is the g-Riesz dual sequence for $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ with respect to $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\gamma_i\}_{i\in\mathbb{N}})$.

We note that if $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}} = \{(K_j^\Lambda)^*\}_{j\in\mathbb{N}}$ is the g-Riesz dual sequence for $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ with respect to $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\eta_i\}_{i\in\mathbb{N}})$, then

$$(\Theta_j^\Lambda)^*(h) = K_j^\Lambda(h) = \sum_{i=1}^\infty \eta_i^* \Lambda_i \gamma_j^* h, \quad h \in H_j, \ j \in \mathbb{N}. \tag{3.1}$$

Throughout this paper, $\{\gamma_j \in B(H, H_j) : j \in \mathbb{N}\}$ and $\{\eta_i \in B(H, H_i) : i \in \mathbb{N}\}$ will be g-orthonormal bases for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. We note that $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}}$ and $\{\Theta_j^\psi\}_{j\in\mathbb{N}}$ are g-Riesz dual sequences for $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\psi = \{\psi_i\}_{i\in\mathbb{N}}$, respectively with respect to $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\eta_i\}_{i\in\mathbb{N}})$.

Extending [1, Lemma 1] we show that we can calculate $\{\Lambda_i\}_{i\in\mathbb{N}}$ by having the g-orthonormal bases $\{\gamma_j\}_{j\in\mathbb{N}}$ and $\{\eta_i\}_{i\in\mathbb{N}}$ and the g-Riesz dual $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}}$.

Lemma 3.3. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$ with Bessel bound B. Then*

$$\Lambda_i^* h = \sum_{j=1}^\infty \gamma_j^* \Theta_j^\Lambda \eta_i^* h, \quad h \in H_i, \ i \in \mathbb{N}.$$

In particular, this shows that $\{\Lambda_i\}_{i\in\mathbb{N}}$ is the g-Riesz dual sequence for $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}}$ with respect to $(\{\eta_i\}_{i\in\mathbb{N}}, \{\gamma_j\}_{j\in\mathbb{N}})$.

Proof. First, we show that $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}}$ is a g-Bessel sequence for H with respect to $\{H_j\}_{j\in\mathbb{N}}$ with Bessel bound B. Since $\{\eta_i\}_{i\in\mathbb{N}}$ and $\{\gamma_j\}_{j\in\mathbb{N}}$ are g-orthonormal bases,

$$\begin{aligned}\sum_{j=1}^\infty \|\Theta_j^\Lambda f\|^2 &= \sum_{j=1}^\infty \left\|\gamma_j\left(\sum_{i=1}^\infty \Lambda_i^* \eta_i f\right)\right\|^2 \\ &= \left\|\sum_{i=1}^\infty \Lambda_i^* \eta_i f\right\|^2 = \|T_\Lambda(\{\eta_i f\}_{i\in\mathbb{N}})\|^2 \\ &\le \|T_\Lambda\|^2 \|\{\eta_i f\}_{i\in\mathbb{N}}\|^2 \le B\|f\|^2, \quad f \in H.\end{aligned}$$

Now, by (3.1), we have

$$\langle k, \Theta_j^\Lambda \eta_i^* h\rangle = \langle (\Theta_j^\Lambda)^* k, \eta_i^* h\rangle = \left\langle \sum_{k=1}^{\infty} \eta_k^* \Lambda_k \gamma_j^* k, \eta_i^* h \right\rangle$$
$$= \langle \Lambda_i \gamma_j^* k, h\rangle = \langle k, \gamma_j \Lambda_i^* h\rangle, \quad h \in H_i, k \in H_j, i, j \in \mathbb{N}.$$

So,

$$\Theta_j^\Lambda \eta_i^* = \gamma_j \Lambda_i^*, \quad i, j \in \mathbb{N}. \tag{3.2}$$

Since $\{\gamma_j\}_{j\in\mathbb{N}}$ is a g-orthonormal basis, by Remark 2.4,

$$\Lambda_i^* h = \sum_{j=1}^{\infty} \gamma_j^* \gamma_j \Lambda_i^* h, \quad h \in H_i, \ i \in \mathbb{N}.$$

Now, by (3.2), $\Lambda_i^* h = \sum_{j=1}^{\infty} \gamma_j^* \Theta_j^\Lambda \eta_i^* h$. □

We note that by Lemma 3.3,

$$\Lambda_i f = \sum_{j=1}^{\infty} \eta_i (\Theta_j^\Lambda)^* \gamma_j f, \quad f \in H, \ i \in \mathbb{N}. \tag{3.3}$$

Remark 3.4. By Lemma 3.3, we see that there is a duality relation between the sequence $\{\Lambda_i\}_{i\in\mathbb{N}}$ and its g-Riesz dual sequence $\{\Theta_j^\Lambda\}_{j\in\mathbb{N}}$, which implies that we can interchange their roles in the results of this section.

In the following theorem we generalize Proposition 5 of [1] to g-frames.

Theorem 3.5. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. Then for all $\{f_i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}} \in (\sum_{i=1}^{\infty} \oplus H_i)_{l_2}$, we have*

$$\left\| \sum_{j=1}^{\infty} (\Theta_j^\Lambda)^* f_j \right\|^2 = \sum_{i=1}^{\infty} \|\Lambda_i f\|^2 \quad \text{and} \quad \left\| \sum_{i=1}^{\infty} \Lambda_i^* g_i \right\|^2 = \sum_{j=1}^{\infty} \|\Theta_j^\Lambda g\|^2,$$

where $f = \sum_{j=1}^{\infty} \gamma_j^* f_j$ and $g = \sum_{i=1}^{\infty} \eta_i^* g_i$.

Proof. Since $\{\eta_i\}_{i\in\mathbb{N}}$ is a g-orthonormal basis and for each $i, j \in \mathbb{N}$, by (3.2), $\Theta_j^\Lambda \eta_i^* = \gamma_j \Lambda_i^*$, we have

$$\sum_{i=1}^{\infty} \|\Lambda_i f\|^2 = \sum_{i=1}^{\infty} \left\| \Lambda_i \left(\sum_{j=1}^{\infty} \gamma_j^* f_j \right) \right\|^2 = \sum_{i=1}^{\infty} \left\| \sum_{j=1}^{\infty} \Lambda_i \gamma_j^* f_j \right\|^2$$
$$= \sum_{i=1}^{\infty} \left\| \sum_{j=1}^{\infty} \eta_i (\Theta_j^\Lambda)^* f_j \right\|^2 = \left\| \sum_{j=1}^{\infty} (\Theta_j^\Lambda)^* f_j \right\|^2.$$

The second claim is similar to the first one. □

Lemma 3.6. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. Suppose that $\{g_i\}_{i\in\mathbb{N}} \in (\sum_{i=1}^{\infty} \oplus H_i)_{l_2}$. Then the following assertions hold:*

(i) $\{g_i\}_{i\in\mathbb{N}} \in \ker T_\Lambda$ *if and only if* $\sum_{i=1}^{\infty} \eta_i^* g_i \in (\operatorname{span}_{j\in\mathbb{N}}\{(\Theta_j^\Lambda)^*(H_j)\})^\perp$.
(ii) $\{g_i\}_{i\in\mathbb{N}} \in \ker T_\Lambda$ *if and only if* $\sum_{i=1}^{\infty} \eta_i^* g_i \in \ker T_{\Theta^\Lambda}^*$.
(iii) *If* $\{g_i\}_{i\in\mathbb{N}} \in R(T_\Lambda^*)$, *then* $\sum_{i=1}^{\infty} \eta_i^* g_i \in R(T_{\Theta^\Lambda})$.

Proof. (i) It follows from (3.2) that, for all $f_j \in H_j$ and $j \in \mathbb{N}$,

$$\left\langle \sum_{i=1}^{\infty} \eta_i^* g_i, (\Theta_j^\Lambda)^* f_j \right\rangle = \left\langle \sum_{i=1}^{\infty} \Lambda_i^* g_i, \gamma_j^* f_j \right\rangle.$$

Applying $\overline{\operatorname{span}}_{j\in\mathbb{N}}\{\gamma_j^*(H_j)\} = H$, we get the result.

(ii) It follows from Theorem 3.5.
(iii) Let $\{g_i\}_{i\in\mathbb{N}} \in R(T_\Lambda^*)$. Then there exists $f \in H$ such that $\{\Lambda_i f\}_{i\in\mathbb{N}} = \{g_i\}_{i\in\mathbb{N}}$. So, by (3.3), we have

$$g_i = \Lambda_i f = \sum_{j=1}^{\infty} \eta_i (\Theta_j^\Lambda)^* \gamma_j f = \eta_i T_{\Theta^\Lambda}(\{\gamma_j f\}_{j\in\mathbb{N}}), \quad i \in \mathbb{N}.$$

Therefore,

$$\eta_i^* g_i = \eta_i^* \eta_i T_{\Theta^\Lambda}(\{\gamma_j f\}_{j\in\mathbb{N}}), \quad i \in \mathbb{N},$$

and by Remark 2.4,

$$\sum_{i=1}^{\infty} \eta_i^* g_i = T_{\Theta^\Lambda}(\{\gamma_j f\}_{j\in\mathbb{N}}).$$

Therefore, $\sum_{i=1}^{\infty} \eta_i^* g_i \in R(T_{\Theta^\Lambda})$. □

In the following proposition by extending Propositions 10 and 11 of [1], we characterize those g-Riesz dual sequences, which can be g-Bessel sequences or sequences with lower frame bounds.

Proposition 3.7. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. Then the following assertions are equivalent:*

(i) $\{\Lambda_i\}_{i\in\mathbb{N}}$ *is a g-frame with bounds A and B.*
(ii) *For each* $\{f_j\}_{j=1}^{\infty} \in (\sum_{j=1}^{\infty} \oplus H_j)_{l_2}$,

$$\sqrt{A}\|\{f_j\}_{j\in\mathbb{N}}\|_2 \le \left\| \sum_{j=1}^{\infty} (\Theta_j^\Lambda)^* f_j \right\| \le \sqrt{B}\|\{f_j\}_{j\in\mathbb{N}}\|_2.$$

Proof. (i) → (ii) Let $\{f_j\}_{j=1}^{\infty} \in (\sum_{j=1}^{\infty} \oplus H_j)_{l_2}$. By Theorem 3.5, we have

$$\left\|\sum_{j=1}^{\infty}(\Theta_j^{\Lambda})^* f_j\right\|^2 = \sum_{i=1}^{\infty} \|\Lambda_i f\|^2,$$

where $f = \sum_{j=1}^{\infty} \gamma_j^* f_j$. Since $\|\{f_j\}_{j\in\mathbb{N}}\|_2^2 = \|f\|^2$, the proof is evident.

(ii) → (i) Since $f = \sum_{j=1}^{\infty} \gamma_j^* \gamma_j f$ and $\{\gamma_j f\}_{j\in\mathbb{N}} \in (\sum_{j=1}^{\infty} \oplus H_j)_{l_2}$, by the assumption we have

$$A\|f\|^2 \leq \left\|\sum_{j=1}^{\infty}(\Theta_j^{\Lambda})^* \gamma_j f\right\|^2 \leq B\|f\|^2.$$

Now, the proof follows from Theorem 3.5. □

Proposition 3.8 ([9]). *Let for each $i \in \mathbb{N}$, $\Lambda_i \in B(H, H_i)$. Then the following assertions are equivalent*:

(i) *$\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-frame sequence.*
(ii) *There exist constants $0 < A \leq B < \infty$ such that for each $f = \{f_i\}_{i\in\mathbb{N}} \in (\ker T_{\Lambda})^{\perp}$,*

$$\sqrt{A}\|f\|_2 \leq \left\|\sum_{i=1}^{\infty} \Lambda_i^* f_i\right\| \leq \sqrt{B}\|f\|_2.$$

The following proposition which is a general version of Proposition 13 of [1] shows a kind of equivalence between a sequence and its g-Riesz dual sequence.

Proposition 3.9. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. Then the following assertions are equivalent*:

(i) *$\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-frame sequence.*
(ii) *$\{\Theta_j^{\Lambda}\}_{j=1}^{\infty}$ is a g-frame sequence.*

Proof. Let $\{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-frame sequence. Then by Proposition 3.8, there exist constants $0 < A \leq B < \infty$ such that for each $f = \{f_i\}_{i\in\mathbb{N}} \in (\ker T_{\Lambda})^{\perp}$,

$$A\|f\|_2^2 \leq \left\|\sum_{i=1}^{\infty} \Lambda_i^* f_i\right\|^2 \leq B\|f\|_2^2. \tag{3.4}$$

Now, let $g \in \overline{\text{span}}_{j\in\mathbb{N}}\{(\Theta_j^{\Lambda})^*(H_j)\}$. By part (i) of Lemma 3.6, since $g = \sum_{i=1}^{\infty} \eta_i^* \eta_i g$ and $\{\eta_i g\}_{i\in\mathbb{N}} \in (\sum_{i=1}^{\infty} \oplus H_i)_{l_2}$, $\{\eta_i g\}_{i\in\mathbb{N}} \in (\ker T_{\Lambda})^{\perp}$. Therefore, by (3.4), we deduce that

$$A\|g\|^2 \leq \left\|\sum_{i=1}^{\infty} \Lambda_i^* \eta_i g\right\|^2 \leq B\|g\|^2.$$

Now, by Theorem 3.5,

$$A\|g\|^2 \leq \sum_{j=1}^{\infty} \|\Theta_j^{\Lambda} g\|^2 \leq B\|g\|^2.$$

This implies that $\{\Theta_j^{\Lambda}\}_{j\in\mathbb{N}}$ is a g-frame sequence.

(ii) → (i) By Remark 3.4, the proof is similar to the first part. □

The next result is a generalization of Proposition 16 of [1] to g-frames.

Proposition 3.10. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. Then the following assertions are equivalent*:

(i) *$\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-Riesz basis.*
(ii) *$\{\Theta_j^{\Lambda}\}_{j\in\mathbb{N}}$ is a g-Riesz basis.*

Proof. (i) → (ii) Suppose that $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-Riesz basis for H. Then by Proposition 3.7, $\{\Theta_j^{\Lambda}\}_{j\in\mathbb{N}}$ is a g-Riesz basis sequence. Since $\{\Lambda_i\}_{i\in\mathbb{N}}$ is a g-Riesz basis, $\ker T_{\Lambda} = \{0\}$. Now, by part (i) of Lemma 3.6, $\{\Theta_j^{\Lambda}\}_{j\in\mathbb{N}}$ is a g-Riesz basis for H.

(ii) → (i) By Remark 3.4, the proof is similar to the first part. □

4. g-Riesz Dual Sequences for Dual g-Frames of $\{\Lambda_i\}_{i\in\mathbb{N}}$

In this section, we try to characterize g-Riesz dual sequences for dual g-frames of $\{\Lambda_i\}_{i\in\mathbb{N}}$ with respect to two g-orthonormal bases $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\eta_i\}_{i\in\mathbb{N}})$. But to this purpose we first need to find an explicit structure for dual g-frames of $\{\Lambda_i\}_{i\in\mathbb{N}}$. In Lemma 4.5 by inspiration of Lemma 4.3, we find this structure.

Let for each $i \in \mathbb{N}$, $\Lambda_i \in B(H, H_i)$. Suppose that for each $i \in \mathbb{N}$, $\{e_{i,j}\}_{j\in\mathbb{N}}$ is an orthonormal basis for H_i. Then the mapping

$$f \mapsto \langle \Lambda_i f, e_{i,j}\rangle,$$

defines a bounded linear functional on H. Consequently, for each $i, j \in \mathbb{N}$, we can find $u_{i,j} \in H$ such that for each $f \in H$, $\langle f, u_{i,j}\rangle = \langle \Lambda_i f, e_{i,j}\rangle$. Hence,

$$\Lambda_i f = \sum_{j=1}^{\infty} \langle f, u_{i,j}\rangle e_{i,j}, \quad f \in H,$$

and

$$\Lambda_i^* h = \sum_{j=1}^{\infty} \langle h, e_{i,j}\rangle u_{i,j}, \quad h \in H_i, \ \ i \in \mathbb{N}.$$

In particular,

$$u_{i,j} = \Lambda_i^* e_{i,j}, \quad i,j \in \mathbb{N}. \tag{4.1}$$

We call $\{u_{i,j}\}_{i,j\in\mathbb{N}}$ the sequence induced by $\{\Lambda_i\}_{i\in\mathbb{N}}$ with respect to $\{e_{i,j}\}_{i,j\in\mathbb{N}}$.

Theorem 4.1 ([8]). *Let for each* $i \in \mathbb{N}$, $\Lambda_i \in B(H, H_i)$. *Suppose that for each* $i, j \in \mathbb{N}$, $u_{i,j}$ *be defined as in* (4.1). *Then the following assertions hold*:

(i) $\{\Lambda_i\}_{i\in\mathbb{N}}$ *is a g-frame* (*respectively, g-Bessel sequence, g-Riesz basis, g-orthonormal basis*) *for* H *if and only if* $\{u_{i,j}\}_{i,j\in\mathbb{N}}$ *is a frame* (*respectively, Bessel sequence, Riesz basis, orthonormal basis*) *for* H.
(ii) *The g-frame operator for* $\{\Lambda_i\}_{i\in\mathbb{N}}$ *coincides with the frame operator for* $\{u_{i,j}\}_{i,j\in\mathbb{N}}$.
(iii) $\{\Lambda_i\}_{i\in\mathbb{N}}$ *and* $\{\tilde{\Lambda}_i\}_{i\in\mathbb{N}}$ *are a pair of* (*canonical*) *dual g-frames if and only if the induced sequences are a pair of* (*canonical*) *dual frames.*

Remark 4.2. Suppose that for each $i \in \mathbb{N}$, $\{e_{i,j}\}_{j\in\mathbb{N}}$ is an orthonormal basis for H_i. For each $i, j \in \mathbb{N}$, we define $E_{i,j} = \{\delta_{i,k} e_{i,j}\}_{k\in\mathbb{N}}$, where $\delta_{i,k}$ is the Kronecker delta. It is clear that $\{E_{i,j}\}_{i,j\in\mathbb{N}}$ is an orthonormal basis for $(\sum_{i=1}^{\infty} \oplus H_i)_{l_2}$ and for each $\{f_k\}_{k\in\mathbb{N}} \in (\sum_{i=1}^{\infty} \oplus H_i)_{l_2}$, we have

$$\langle \{f_k\}_{k\in\mathbb{N}}, E_{i,j}\rangle = \langle f_i, e_{i,j}\rangle.$$

The space $l^2(\mathbb{N}\times\mathbb{N})$ defined by

$$l^2(\mathbb{N}\times\mathbb{N}) = \left\{ \{a_{i,j}\}_{i,j\in\mathbb{N}} : a_{i,j} \in \mathbb{C}, \sum_{i,j=1}^{\infty} |a_{i,j}|^2 < \infty \right\},$$

with inner product given by

$$\langle \{a_{i,j}\}_{i,j\in\mathbb{N}}, \{b_{i,j}\}_{i,j\in\mathbb{N}}\rangle = \sum_{i,j=1}^{\infty} \langle a_{i,j}, b_{i,j}\rangle,$$

is a Hilbert space.

We write every $a = \{a_{i,j}\}_{i,j\in\mathbb{N}} \in l^2(\mathbb{N}\times\mathbb{N})$ in the following form:

$$a = (\underbrace{a_{1,1}, a_{1,2}, \ldots,}\underbrace{a_{2,1}, a_{2,2}, \ldots,}\underbrace{a_{n,1}, a_{n,2}, \ldots,}\ldots).$$

For each $i, j \in \mathbb{N}$, we define $\alpha_{i,j} = \{b_{m,n}\}_{m,n\in\mathbb{N}}$, where $b_{m,n} = 1$, if $m = i, n = j$ and otherwise $b_{m,n} = 0$. Then $\{\alpha_{i,j}\}_{i,j\in\mathbb{N}}$ is an orthonormal basis for $l^2(\mathbb{N}\times\mathbb{N})$; it is called the canonical orthonormal basis for $l^2(\mathbb{N}\times\mathbb{N})$.

We now aim at a characterization of all g-Riesz dual sequences associated to a given dual g-frame of $\{\Lambda_i\}_{i\in\mathbb{N}}$ with respect to two g-orthonormal bases $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\eta_i\}_{i\in\mathbb{N}})$. To prove our results we will use the following lemmas.

Lemma 4.3 ([2]). *Let* $\{f_i\}_{i\in\mathbb{N}}$ *be a frame for* H *and* $\{\delta_i\}_{i\in\mathbb{N}}$ *be the canonical orthonormal basis for* $l^2(\mathbb{N})$. *The dual frames for* $\{f_i\}_{i\in\mathbb{N}}$ *are precisely the sequences* $\{g_i\}_{i\in\mathbb{N}} = \{V\delta_i\}_{i\in\mathbb{N}}$, *where* $V : l^2(\mathbb{N}) \to H$ *is a bounded left inverse of* $T^*_{\{f_i\}_{i\in\mathbb{N}}}$ (*the analysis operator of* $\{f_i\}_{i\in\mathbb{N}}$).

Lemma 4.4 ([5]). *Let $\{f_k\}_{k\in\mathbb{N}}$ be a frame for H with synthesis operator $T_{\{f_k\}_{k\in\mathbb{N}}}$. The bounded left inverses of $T^*_{\{f_k\}_{k\in\mathbb{N}}}$ are precisely the operators having the form $S^{-1}T_{\{f_k\}_{k\in\mathbb{N}}} + W(I - T^*_{\{f_k\}_{k\in\mathbb{N}}}S^{-1}T_{\{f_k\}_{k\in\mathbb{N}}})$, where $W : l^2(\mathbb{N}) \to H$ is a bounded operator and S is the frame operator for $\{f_k\}_{k\in\mathbb{N}}$.*

We begin with a lemma, which characterizes a general form of dual g-frames of $\{\Lambda_i\}_{i\in\mathbb{N}}$.

Lemma 4.5. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-frame for H with respect to $\{H_i\}_{i\in\mathbb{N}}$ and for each $i \in \mathbb{N}$, $\{e_{i,j}\}_{j\in\mathbb{N}}$ be an orthonormal basis for H_i. The dual g-frames of $\{\Lambda_i\}_{i\in\mathbb{N}}$ are precisely the sequences $\psi = \{\psi_i\}_{i\in\mathbb{N}} = \{\phi_i^* V^*\}_{i\in\mathbb{N}}$, where $V : l^2(\mathbb{N}\times\mathbb{N}) \to H$, is a bounded left inverse of the analysis operator of the induced frame $\{u_{i,j}\}_{i,j\in\mathbb{N}}$ and for each $i \in \mathbb{N}$, $\phi_i : H_i \to l^2(\mathbb{N}\times\mathbb{N})$ is an isometric isomorphism of H_i onto a subspace of $l^2(\mathbb{N}\times\mathbb{N})$.*

Proof. Assume that $\{\psi_i\}_{i\in\mathbb{N}}$ is a dual g-frame of $\{\Lambda_i\}_{i\in\mathbb{N}}$. Then by Theorem 4.1, $\{v_{i,j}\}_{i,j\in\mathbb{N}} = \{\psi_i^* e_{i,j}\}_{i,j\in\mathbb{N}}$ is a dual frame of $\{u_{i,j}\}_{i,j\in\mathbb{N}} = \{\Lambda_i^* e_{i,j}\}_{i,j\in\mathbb{N}}$. By Lemma 4.3,

$$\{v_{i,j}\}_{i,j\in\mathbb{N}} = \{V\alpha_{i,j}\}_{i,j\in\mathbb{N}}, \tag{4.2}$$

where $V : l^2(\mathbb{N}\times\mathbb{N}) \to H$ is a bounded left inverse of T^* (the analysis operator of the frame $\{u_{i,j}\}_{i,j\in\mathbb{N}}$), and $\{\alpha_{i,j}\}_{i,j\in\mathbb{N}}$ is the canonical orthonormal basis for $l^2(\mathbb{N}\times\mathbb{N})$. We define the mapping

$$\phi_i : H_i \to l^2(\mathbb{N}\times\mathbb{N}), \quad \phi_i\left(\sum_{j=1}^{\infty} c_{i,j}e_{i,j}\right) = \sum_{j=1}^{\infty} c_{i,j}\alpha_{i,j}, \ \ i \in \mathbb{N}. \tag{4.3}$$

Clearly, the mapping ϕ_i is well defined and is an isometric isomorphism of H_i onto a subspace of $l^2(\mathbb{N}\times\mathbb{N})$. Since $\{e_{i,j}\}_{j\in\mathbb{N}}$ is an orthonormal basis for H_i, by (4.2) and (4.3), we have

$$\begin{aligned}\psi_i^*(h) &= \psi_i^*\left(\sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle e_{i,j}\right) = \sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle v_{i,j} = \sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle V(\alpha_{i,j})\\ &= V\left(\sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle \alpha_{i,j}\right) = V\left(\sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle \phi_i(e_{i,j})\right)\\ &= V\phi_i\left(\sum_{j=1}^{\infty}\langle h, e_{i,j}\rangle e_{i,j}\right) = V\phi_i(h), \quad h \in H_i, \ \ i \in \mathbb{N}.\end{aligned}$$

So, for each $i \in \mathbb{N}$, $\psi_i = \phi_i^* V^*$.

Now, we show that $\{\psi_i\}_{i\in\mathbb{N}}=\{\phi_i^* V^*\}_{i\in\mathbb{N}}$ is a dual g-frame of $\{\Lambda_i\}_{i\in\mathbb{N}}$. We define

$$T_\psi : \left(\sum_{i=1}^{\infty} \oplus H_i\right)_{l_2} \to H, \quad T_\psi(\{g_i\}_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty} \psi_i^* g_i.$$

Since for each $i \in \mathbb{N}$, $\psi_i = \phi_i^* V^*$, by (4.3), we have

$$\begin{aligned} T_\psi(\{g_i\}_{i\in\mathbb{N}}) &= \sum_{i=1}^{\infty} \psi_i^* g_i = \sum_{i=1}^{\infty} V\phi_i g_i = V\left(\sum_{i=1}^{\infty} \phi_i g_i\right) \\ &= V\left(\sum_{i=1}^{\infty} \phi_i\left(\sum_{j=1}^{\infty} \langle g_i, e_{i,j}\rangle e_{i,j}\right)\right) = V\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle g_i, e_{i,j}\rangle \alpha_{i,j}\right). \quad (4.4) \end{aligned}$$

We define the mapping

$$\beta : \left(\sum_{i=1}^{\infty} \oplus H_i\right)_{l_2} \to l^2(\mathbb{N}\times\mathbb{N}), \quad \beta\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j} E_{i,j}\right) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} c_{i,j}\alpha_{i,j}, \qquad (4.5)$$

where $\{E_{i,j}\}_{i,j\in\mathbb{N}}$ is an orthonormal basis for $(\sum_{i=1}^{\infty} \oplus H_i)_{l_2}$. Clearly, β is a well defined and isometric isomorphism operator. So, by (4.4), (4.5) and Remark 4.2, we have

$$T_\psi(\{g_i\}_{i\in\mathbb{N}}) = V\beta\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle g_i, e_{i,j}\rangle E_{i,j}\right) = V\beta(\{g_i\}_{i\in\mathbb{N}}).$$

Therefore, $T_\psi = V\beta$. Since V is a bounded left inverse of T^*, V is surjective and so T_ψ is a well defined, bounded and surjective operator from $(\sum_{i=1}^{\infty} \oplus H_i)_{l_2}$ onto H. Hence, by [6, Proposition 2.6], $\psi = \{\psi_i\}_{i\in\mathbb{N}}$ is a g-frame for H with respect to $\{H_i\}_{i\in\mathbb{N}}$.

Since V is a bounded left inverse of T^* (the analysis operator of $\{u_{i,j}\}_{i,j\in\mathbb{N}}$), we have

$$\begin{aligned} f = VT^*f &= V(\{\langle f, u_{i,j}\rangle\}_{i,j\in\mathbb{N}}) = V(\{\langle f, \Lambda_i^* e_{i,j}\rangle\}_{i,j\in\mathbb{N}}) \\ &= V(\{\langle \Lambda_i f, e_{i,j}\rangle\}_{i,j\in\mathbb{N}}), \quad f \in H. \quad (4.6) \end{aligned}$$

Since $T_\psi = V\beta$, by (4.5), (4.6) and Remark 4.2,

$$\begin{aligned} f &= T_\psi\beta^{-1}(\{\langle \Lambda_i f, e_{i,j}\rangle\}_{i,j\in\mathbb{N}}) = T_\psi\beta^{-1}\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle \Lambda_i f, e_{i,j}\rangle \alpha_{i,j}\right) \\ &= T_\psi\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle \Lambda_i f, e_{i,j}\rangle E_{i,j}\right) = T_\psi(\{\Lambda_i f\}_{i\in\mathbb{N}}) = T_\psi T_\Lambda^* f, \quad f \in H. \quad \square \end{aligned}$$

Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-frame for H with respect to $\{H_i\}_{i\in\mathbb{N}}$ with g-frame operator S_Λ. Then we define

$$(\Theta_j^{\Lambda S_\Lambda^{-1}})^*(h) = (\Theta_j^{\tilde{\Lambda}})^*(h) = \sum_{i=1}^{\infty} \eta_i^* \Lambda_i S_\Lambda^{-1} \gamma_j^* h, \quad h \in H_j,\ j \in \mathbb{N}.$$

This means that $\{\Theta_j^{\tilde{\Lambda}}\}_{j\in\mathbb{N}}$ is the g-Riesz dual sequence for $\Lambda S_\Lambda^{-1} = \{\Lambda_i S_\Lambda^{-1}\}_{i\in\mathbb{N}}$ with respect to $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\eta_i\}_{i\in\mathbb{N}})$.

We are now ready for the announced characterization of all g-Riesz dual sequences associated to a given dual g-frame of $\{\Lambda_i\}_{i\in\mathbb{N}}$ with respect to two orthonormal bases $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\eta_i\}_{i\in\mathbb{N}})$.

Theorem 4.6. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-frame for H with respect to $\{H_i\}_{i\in\mathbb{N}}$ with g-frame operator S_Λ. Suppose that $\psi = \{\psi_i\}_{i\in\mathbb{N}}$ is a dual g-frame of $\{\Lambda_i\}_{i\in\mathbb{N}}$. Then*

$$(\Theta_j^{\psi})^*(h) = (\Theta_j^{\tilde{\Lambda}})^*(h) + (\Theta_j^{\phi^* W^*})^*(h) - (\Theta_j^{\tilde{\Lambda} T W^*})^*(h), \quad h \in H_j,\ j \in \mathbb{N},$$

where $\{\Theta_j^{\tilde{\Lambda}}\}_{j\in\mathbb{N}}$, $\{\Theta_j^{\phi^* W^*}\}_{j\in\mathbb{N}}$ and $\{\Theta_j^{\tilde{\Lambda} T W^*}\}_{j\in\mathbb{N}}$ are g-Riesz dual sequences for $\{\tilde{\Lambda}_i\}_{i\in\mathbb{N}}$, $\{\phi_i^* W^*\}_{i\in\mathbb{N}}$ and $\{\tilde{\Lambda}_i T W^*\}_{i\in\mathbb{N}}$, respectively with respect to $(\{\gamma_j\}_{j\in\mathbb{N}}, \{\eta_i\}_{i\in\mathbb{N}})$.

Proof. Suppose that $\psi = \{\psi_i\}_{i\in\mathbb{N}}$ is a dual g-frame of $\{\Lambda_i\}_{i\in\mathbb{N}}$. By Lemma 4.5,

$$\psi_i = \phi_i^* V^*, \quad i \in \mathbb{N}, \tag{4.7}$$

where V is a bounded left inverse of T^* (the analysis operator of $\{u_{i,j}\}_{i,j\in\mathbb{N}}$), and ϕ_i is the mapping defined in (4.3). Since the g-frame operator for $\{\Lambda_i\}_{i\in\mathbb{N}}$ coincides with the frame operator for $\{u_{i,j}\}_{i,j\in\mathbb{N}}$, by Lemma 4.4, we have

$$V = S_\Lambda^{-1} T + W(I - T^* S_\Lambda^{-1} T),$$

where $W : l^2(\mathbb{N}\times\mathbb{N}) \to H$ is a bounded operator and T is the synthesis operator of the frame $\{u_{i,j}\}_{i,j\in\mathbb{N}}$. So,

$$V^* = T^* S_\Lambda^{-1} + (I - T^* S_\Lambda^{-1} T) W^*. \tag{4.8}$$

Therefore, by (3.1), (4.7) and (4.8), we have

$$\begin{aligned}
(\Theta_j^{\psi})^*(h) &= \sum_{i=1}^{\infty} \eta_i^* \psi_i \gamma_j^* h = \sum_{i=1}^{\infty} \eta_i^* \phi_i^* V^* \gamma_j^* h \\
&= \sum_{i=1}^{\infty} \eta_i^* \phi_i^* (T^* S_\Lambda^{-1} + (I - T^* S_\Lambda^{-1} T) W^*) \gamma_j^* h \\
&= \sum_{i=1}^{\infty} \eta_i^* \phi_i^* T^* S_\Lambda^{-1} \gamma_j^* h + \sum_{i=1}^{\infty} \eta_i^* \phi_i^* W^* \gamma_j^* h \\
&\quad - \sum_{i=1}^{\infty} \eta_i^* \phi_i^* T^* S_\Lambda^{-1} T W^* \gamma_j^* h, \quad h \in H_j,\ j \in \mathbb{N}.
\end{aligned} \tag{4.9}$$

Since $\{e_{i,j}\}_{j\in\mathbb{N}}$ is an orthonormal basis for H_i and $\{\alpha_{i,j}\}_{i,j\in\mathbb{N}}$ is the canonical orthonormal basis for $l^2(\mathbb{N}\times\mathbb{N})$, by (4.3), we have

$$T\phi_i(e_{i,j}) = T(\alpha_{i,j}) = u_{i,j} = \Lambda_i^*(e_{i,j}), \quad i,j\in\mathbb{N}.$$

So,

$$T\phi_i = \Lambda_i^*, \quad i\in\mathbb{N}. \tag{4.10}$$

Hence, by (4.9) and (4.10),

$$(\Theta_j^{\psi})^*(h) = \sum_{i=1}^{\infty}\eta_i^*\tilde{\Lambda}_i\gamma_j^*h + \sum_{i=1}^{\infty}\eta_i^*\phi_i^*W^*\gamma_j^*h - \sum_{i=1}^{\infty}\eta_i^*\tilde{\Lambda}_iTW^*\gamma_j^*h, \quad h\in H_j,\ \ j\in\mathbb{N}.$$

Now, we show that $\{\phi_i^*W^*\}_{i\in\mathbb{N}}$ is a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$.

We define the mapping

$$K:\left(\sum_{i=1}^{\infty}\oplus H_i\right)_{l_2}\to H, \quad K(\{f_i\}_{i\in\mathbb{N}}) = \sum_{i=1}^{\infty}W\phi_if_i.$$

Since $\{e_{i,j}\}_{j\in\mathbb{N}}$ is an orthonormal basis for H_i, by (4.3), (4.5) and Remark 4.2, we have

$$\begin{aligned}K(\{f_i\}_{i\in\mathbb{N}}) &= \sum_{i=1}^{\infty}W\phi_if_i = W\left(\sum_{i=1}^{\infty}\phi_if_i\right) = W\left(\sum_{i=1}^{\infty}\phi_i\left(\sum_{j=1}^{\infty}\langle f_i, e_{i,j}\rangle e_{i,j}\right)\right)\\ &= W\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle f_i, e_{i,j}\rangle\alpha_{i,j}\right) = W\beta\left(\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\langle f_i, e_{i,j}\rangle E_{i,j}\right)\\ &= W\beta(\{f_i\}).\end{aligned}$$

Therefore, K is a well defined and bounded operator of $(\sum_{i=1}^{\infty}\oplus H_i)_{l_2}$ into H and by [6, Proposition 2.4], $\{\phi_i^*W^*\}_{i\in\mathbb{N}}$ is a g-Bessel sequence for H with respect to $\{H_i\}_{i\in\mathbb{N}}$. So, we have

$$(\Theta_j^{\psi})^*(h) = (\Theta_j^{\tilde{\Lambda}})^*(h) + (\Theta_j^{\phi^*W^*})^*(h) - (\Theta_j^{\tilde{\Lambda}TW^*})^*(h), \quad h\in H_j,\ \ j\in\mathbb{N}. \qquad \square$$

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