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On the completeness of normed spaces

Abbas Najati

Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil, Iran

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ABSTRACT

Let $(G, +)$ be an abelian group and let E be a normed space. A mapping $f : G \to E$ is called ε-quadratic if for a given $\varepsilon > 0$ it satisfies $\|f(x+y)+f(x-y)-2f(x)-2f(y)\| \le \varepsilon$ for all $x, y \in G$. In this work, we show that E is complete if every ε-quadratic mapping $f : X \to E$ can be estimated by a quadratic mapping, where X is $\mathbb{N}_0$ or a finitely generated free abelian group.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms: *Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality*

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces: *Assume that $f : X \to Y$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for some $\varepsilon \ge 0$ and all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$.

Schwaiger [3] proved that a normed space E is complete if for each $f : \mathbb{Z} \to E$ whose Cauchy difference $f(x+y) - f(x) - f(y)$ is bounded for all $x, y \in \mathbb{Z}$, there exists an additive mapping $T : \mathbb{Z} \to E$ such that $f - T$ is uniformly bounded on $\mathbb{Z}$. Later, Forti and Schwaiger [4] proved that Schwaiger's theorem remains true if we replace $\mathbb{Z}$ by any abelian group containing an element of infinite order. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a *quadratic functional equation*. Quadratic functional equations were used to characterize inner product spaces [5–7]. In particular, every solution of the quadratic equation (1.1) is said to be a *quadratic mapping*. It is well known that a mapping

E-mail address: a.nejati@yahoo.com.

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f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive mapping B such that $f(x) = B(x, x)$ for all x (see [5,8]). The bi-additive mapping B is given by

$$B(x, y) = \frac{1}{4}[f(x + y) - f(x - y)].$$

The generalized Hyers–Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space (see [9]). Cholewa [10] noticed that the theorem of Skof is still true if E_1 is replaced by an abelian group. In [11], Czerwik proved the generalized Hyers–Ulam stability of the quadratic functional equation (1.1). Grabiec [12] has generalized these results mentioned above. Jun and Lee [13] proved the generalized Hyers–Ulam stability of a Pexiderized quadratic equation.

Let E be a normed space. We denote the set of non-negative integers and the set of integers by $\mathbb{N}_0$ and $\mathbb{Z}$, respectively. In this work, we show that E is complete if every ε-quadratic mapping $f : X \to E$ can be estimated by a quadratic mapping, where X is $\mathbb{N}_0$ or a finitely generated free abelian group.

2. Main results

Let us we denote the sets $\underbrace{\mathbb{N}_0 \times \mathbb{N}_0 \times \cdots \times \mathbb{N}_0}_{r\text{-times}}$ and $\underbrace{\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}}_{r\text{-times}}$ by $\mathbb{N}_0^r$ and $\mathbb{Z}^r$, respectively. The proof of the following lemma is obvious:

Lemma 2.1. *Let $f : \mathbb{N}_0^r \to E$ $(f : \mathbb{Z}^r \to E)$ be a quadratic mapping, i.e.,*

$$f(m_1 + n_1, \ldots, m_r + n_r) + f(m_1 - n_1, \ldots, m_r - n_r) = 2f(m_1, \ldots, m_r) + 2f(n_1, \ldots, n_r)$$

for all $m_i \ge n_i \ge 0$ $(m_i, n_i \in \mathbb{Z})$ with $1 \le i \le r$. Then

$$f(m, \ldots, m) = m^2 f(1, \ldots, 1)$$

for all $m \in \mathbb{N}_0$ $(m \in \mathbb{Z})$.

Lemma 2.2. *Let $f : \mathbb{N}_0 \to E$ be a ε-quadratic mapping, i.e.,*

$$\|f(m + n) + f(m - n) - 2f(m) - 2f(n)\| \le \varepsilon \quad (m \ge n \ge 0).$$

Then the mapping $g : \mathbb{Z} \to E$ defined by $g(n) = f(|n|)$ is ε-quadratic.

Proof. For convenience, we use the following abbreviations:

$$Q\,f(m, n) := f(m + n) + f(m - n) - 2f(m) - 2f(n) \quad (m \ge n \ge 0),$$
$$Q\,g(m, n) := g(m + n) + g(m - n) - 2g(m) - 2g(n) \quad (m, n \in \mathbb{Z}).$$

It follows from the definition of g that

$$Q\,g(m, n) = \begin{cases} Q\,f(m, n), & \text{if } m \ge n \ge 0; \\ Q\,f(n, m), & \text{if } m, n \ge 0, \ \ m < n; \\ Q\,f(-n, -m), & \text{if } m, n < 0, \ \ m \ge n; \\ Q\,f(-m, -n), & \text{if } m, n < 0, \ \ m < n; \\ Q\,f(m, -n), & \text{if } n < 0 \le m, \ \ m + n \ge 0; \\ Q\,f(-n, m), & \text{if } n < 0 \le m, \ \ m + n < 0; \\ Q\,f(n, -m), & \text{if } m < 0 \le n, \ \ m + n \ge 0; \\ Q\,f(-m, n), & \text{if } m < 0 \le n, \ \ m + n < 0. \end{cases}$$

Indeed,

$$Q\,g(m, n) = \begin{cases} Q\,f(|m|, |n|), & \text{if } |m| \ge |n|; \\ Q\,f(|n|, |m|), & \text{if } |n| \ge |m| \end{cases}$$

for all $m, n \in \mathbb{Z}$. Therefore g is ε-quadratic on $\mathbb{Z}$. □

Using the proof of Lemma 2.2, we have

Proposition 2.3. *Let $f : \mathbb{N}_0^r \to E$ be a ε-quadratic mapping. Then the mapping $g : \mathbb{Z}^r \to E$ defined by*

$$g(n_1, n_2, \ldots, n_r) = f(|n_1|, |n_2|, \ldots, |n_r|)$$

is ε-quadratic.

Theorem 2.4. *Let E be a normed space such that for each ε-quadratic mapping $f : \mathbb{Z}^r \to E$ there exists a quadratic mapping $Q : \mathbb{Z}^r \to E$ such that $Q - f$ is uniformly bounded on $\mathbb{Z}^r$. Then E is complete.*

Proof. Let us have $\varepsilon > 0$ and let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in E. There exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ such that $\|x_n - x_m\| < \frac{\varepsilon}{6k^2}$ for all $m, n \geq n_k$. Let $y_0 = 0$ and $y_k = x_{n_k}$ for all $k \geq 1$. We define the mapping $f : \mathbb{N}_0^r \to E$ by $f(k_1, \ldots, k_r) = \frac{1}{r}\sum_{i=1}^{r} k_i^2 y_{k_i}$ for all $(k_1, \ldots, k_r) \in \mathbb{N}_0^r$. Then

$$\begin{aligned}
&\|f(m_1 + k_1, \ldots, m_r + k_r) + f(m_1 - k_1, \ldots, m_r - k_r) - 2f(m_1, \ldots, m_r) - 2f(k_1, \ldots, k_r)\| \\
&\quad = \frac{1}{r}\left\|\sum_{i=1}^{r}[(m_i^2 + 2m_ik_i)(y_{m_i+k_i} - y_{m_i}) + k_i^2(y_{m_i+k_i} - y_{k_i}) + (m_i - k_i)^2(y_{m_i-k_i} - y_{m_i}) + k_i^2(y_{m_i} - y_{k_i})]\right\| \\
&\quad \leq \frac{1}{r}\sum_{i=1}^{r}[(m_i^2 + 2m_ik_i)\|y_{m_i+k_i} - y_{m_i}\| + k_i^2\|y_{m_i+k_i} - y_{k_i}\| + (m_i - k_i)^2\|y_{m_i-k_i} - y_{m_i}\| + k_i^2\|y_{m_i} - y_{k_i}\|] \\
&\quad \leq \frac{1}{r}\sum_{i=1}^{r}\left[\frac{\varepsilon}{6} + \frac{2k_i\varepsilon}{6m_i} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6}\right] \leq \varepsilon
\end{aligned}$$

for all $m_i > k_i \geq 0$, where $1 \leq i \leq r$. Hence it follows from $f(0, \ldots, 0) = 0$ that f is ε-quadratic. By Lemma 2.2, the mapping $g : \mathbb{Z}^r \to E$ defined by $g(k_1, \ldots, k_r) = f(|k_1|, \ldots, |k_r|)$ is ε-quadratic. By our assumption, there exists a quadratic mapping $Q : \mathbb{Z}^r \to E$ and a positive constant M such that $\|g(k_1, \ldots, k_r) - Q(k_1, \ldots, k_r)\| \leq M$ for all $(k_1, \ldots, k_r) \in \mathbb{Z}^r$. Since Q is quadratic on $\mathbb{Z}^r$, by Lemma 2.1 we have $Q(k, \ldots, k) = k^2 Q(1, \ldots, 1)$ for all $k \in \mathbb{Z}$. Therefore $\|k^2 x_{n_k} - k^2 Q(1, \ldots, 1)\| \leq M$ for all positive integers k. This shows that the subsequence $\{x_{n_k}\}_{k\geq 1}$ converges to $Q(1, \ldots, 1)$. Hence the Cauchy sequence $\{x_n\}_{n\geq 1}$ also converges to $Q(1, \ldots, 1)$, and the theorem is proved. □

The following result follows from the proof of Theorem 2.4.

Theorem 2.5. *Let E be a normed space such that for each ε-quadratic mapping $f : \mathbb{N}_0^r \to E$ there exists a quadratic mapping $Q : \mathbb{N}_0^r \to E$ such that $Q - f$ is uniformly bounded on $\mathbb{N}_0^r$. Then E is complete.*

Corollary 2.6. *Let E be a normed space such that for each ε-quadratic mapping $f : \mathbb{Z} \to E$ ($f : \mathbb{N}_0 \to E$) there exists a quadratic mapping $Q : \mathbb{Z} \to E$ ($Q : \mathbb{N}_0 \to E$) such that $Q - f$ is uniformly bounded on $\mathbb{Z}$ ($\mathbb{N}_0$). Then E is complete.*

Theorem 2.4 remains true if we replace $\mathbb{Z}^r$ by a finitely generated free abelian group.

Theorem 2.7. *Let G be a finitely generated free abelian group and E be a normed space such that for each ε-quadratic mapping $f : G \to E$ there exists a quadratic mapping $Q : G \to E$ such that $Q - f$ is uniformly bounded on G. Then E is complete.*

Proof. Let $\{g_1, \ldots, g_r\}$ be a basis for G and let $f : \mathbb{Z}^r \to E$ be a ε-quadratic mapping. We define the mapping $\varphi : G \to E$ by $\varphi(\sum_{i=1}^{r} n_i g_i) = f(n_1, \ldots, n_r)$. It is clear that φ is ε-quadratic. By the assumption, there exists a quadratic mapping $Q : G \to E$ such that $Q - \varphi$ is uniformly bounded on G. Let $T : \mathbb{Z}^r \to E$ be a mapping defined by $T(n_1, \ldots, n_r) = Q(\sum_{i=1}^{r} n_i g_i)$. Hence T is quadratic and $T - f$ is uniformly bounded on $\mathbb{Z}^r$. Hence by Theorem 2.4, E is complete. □

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