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C-FRAMES AND C-BESSEL MAPPINGS

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Abstract. The theory of C-frames and C-Bessel mappings are the generalizations of the theory of frames and Bessel sequences. In this article, we obtain several equivalence conditions for duals of C-Bessel mappings. We show that for a C-Bessel mapping f, an retrieval formula with respect to a C-Bessel mapping g is satisfied if and only if g is sum of the canonical dual of f with a C-Bessel mapping which is weakly in the null space of the pre-frame operator of f. Also, we prove that composition of pre-frame operator with analysis operator of two square norm integrable C-Bessel mappings are trace class operators.

1. Introduction

Today, the theory of discrete and continuous frames play important roles not just in digital signal processing and scientific computations but also in both pure and applied mathematics. The notion of frames was first introduced by Duffin and Shaeffer [3]in connection with some problems in non-harmonic analysis. Throughout this paper (X, μ) will be a measure space and H will be a Hilbert space over $\mathbb{C}$. We shall denote the closed unit ball of H by H_1.

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Definition 1.1. *Let $L^2(X,H)$ be the class of all measurable mappings $f : X \to H$ such that*

$$||f||_2^2 = \int_X ||f(x)||^2 d\mu < \infty.$$

By the polar identity, we conclude that for each $f, g \in L^2(X,H)$, the mapping $x \mapsto \langle f(x), g(x)\rangle$ of X to $\mathbb{C}$ is measurable, and it can be proved that $L^2(X,H)$ is a Hilbert space with the inner product defined by

$$\langle f, g\rangle_{L^2} = \int_X \langle f(x), g(x)\rangle d\mu.$$

We shall write $L^2(X)$ when $H = \mathbb{C}$.

The following Lemmas can be found in operator theory text books.

Lemma 1.2. *Let $u : K \to H$ be a bounded operator with closed range $\mathcal{R}_u$. Then there exists a bounded operator $u^\dagger : H \to K$ for which*

$$uu^\dagger f = f, \quad f \in \mathcal{R}_u.$$

Also, $u^* : H \to K$ has closed range and $(u^*)^\dagger = (u^\dagger)^*$.

Lemma 1.3. *Let $u : H \to K$ be a bounded operator. Then:*
(i) $||u|| = ||u^*||$ and $||uu^*|| = ||u||^2$.
(ii) $\mathcal{R}_u$ is closed if and only if $\mathcal{R}_{u^*}$ is closed.
(iii) u is surjective if and only if there exists $c > 0$ such that for each $h \in H$

$$c||h|| \leq ||u^*h||.$$

Lemma 1.4. *Let $u : K \to H$ be a bounded surjective operator. Given $y \in H$, the equation $ux = y$ has a unique solution of minimal norm, namely, $x = u^\dagger y$.*

Lemma 1.5. *Let u be a self-adjoint bounded operator on H. Let*

$$m_u = \inf_{h \in H_1} \langle uh, h\rangle \quad \text{and} \quad M_u = \sup_{h \in H_1} \langle uh, h\rangle.$$

Then, $m_u, M_u \in \sigma(u)$.

2. C-frames

Definition 2.1. *Let $f : X \to H$ be a mapping. We say that f is weakly measurable if for each $h \in H$ the mapping $x \mapsto \langle h, f(x)\rangle$ of X to $\mathbb{C}$ is measurable.*

Each Bochner measurable mapping $f : X \to H$ is weakly measurable, indeed.

Definition 2.2. *Let $f : X \to H$ be weakly measurable. We say that f is a C-frame for H, if there exist $A, B > 0$ such that*

$$A||h||^2 \leq \int_X |\langle h, f(x)\rangle|^2 d\mu \leq B||h||^2, \quad h \in H.$$

If only the right hand inequality is satisfied then we say that f is a C-Bessel mapping for H.

Let $f : X \to H$ be a C-Bessel mapping for H. Let $T_f : L^2(X) \to H$ be defined by

$$\langle T_f(g), h\rangle = \int_X g(x)\langle f(x), h\rangle d\mu, \quad h \in H, g \in L^2(X).$$

It is evident that T_f is well defined and linear. For each $g \in L^2(X)$ and $h \in H$,

$$||T_f(g)|| = \sup_{h\in H_1} |\langle T_f(g), h\rangle|$$

$$\leq (\int_X |g(x)|^2 d\mu)^{1/2} \sup_{h\in H_1} (\int_X |\langle f(x), h\rangle|^2 d\mu)^{1/2} \leq B||g||_2.$$

Hence, T_f is bounded. We shall denote $T_f : L^2(X) \to H$ by

$$T_f(g) = \int_X gf d\mu, \quad g \in L^2(X),$$

and we call it the pre-frame operator of f.
For each $g \in L^2(X)$ and $h \in H$ we have

$$\langle T_f^*(h), g\rangle = \langle h, T_f(g)\rangle = \overline{\langle T_f(g), h\rangle}$$

$$= \int_X \overline{g(x)}\langle h, f(x)\rangle d\mu = \langle\langle h, f\rangle, g\rangle.$$

Thus

$$T_f^*(h) = \langle h, f \rangle.$$

Also, for each $h \in H$

$$||T_f^*(h)||^2 = \langle T_f^*(h), T_f^*(h) \rangle = \int_X |\langle f(x), h \rangle|^2 d\mu.$$

Therefore

$$||T_f|| = ||T_f^*|| = (\sup_{h \in H_1} \int_X |\langle f(x), h \rangle|^2 d\mu)^{1/2}.$$

The mapping $T_f^* : H \to L^2(X)$ is called the analysis operator of f.
We define, $S_f : H \to H$ by

$$S_f(h) = T_f T_f^*(h) = T_f(\langle h, f \rangle) = \int_X \langle h, f \rangle f d\mu,$$

and we call it the frame operator of f.

It is clear that each element of $L^2(X, H)$ is a C-Bessel mapping for H.

Lemma 2.3. *Let H be a Hilbert space. Then:*
(i) If $\dim H < \infty$ then $L^2(X, H)$ is the class of all C-Bessel mappings for H.
(ii) Let μ be a σ-finite measure. If there exists $f \in L^2(X, H)$ such that f is a C-frame for H then

$$dim H < \infty.$$

Proof. Let $\{e_\alpha\}_{\alpha \in I}$ be an orthonormal basis for H. Since, for each mapping $f : X \to H$

$$\int_X ||f(x)||^2 d\mu = \int_X \sum_\alpha |\langle f(x), e_\alpha \rangle|^2 d\mu = \sum_\alpha \int_X |\langle f(x), e_\alpha \rangle|^2 d\mu,$$

the proof is evident.

Lemma 2.4. *Let $f : X \to H$ be a C-Bessel mapping for H with upper bound B . Then, the following assertions are equivalent:*
(i) The frame operator $S_f : H \to H$ is invertible.
(ii) The pre-frame operator $T_f : L^2(X) \to H$ is surjective.

Proof. $(i) \Rightarrow (ii)$ Let S_f be invertible. By the Lemma (1.5), we have

$$\inf_{h\in H_1} ||T_f^*(h)||^2 = \inf_{h\in H_1} \langle T_f T_f^*(h), h\rangle \in \sigma(S_f).$$

So, $\inf_{h\in H_1} ||T_f^*(h)|| > 0$. Since

$$\inf_{h\in H_1} ||T_f^*(h)||.||h|| \leq ||T_f^*(h)||,$$

by the Lemma (1.3)(iii), T_f is surjective.
$(ii) \to (i)$ Let T_f be surjective. Then, there exists $A > 0$ such that

$$A||h||^2 \leq ||T_f^*(h)||^2, \quad h \in H.$$

Therefore,

$$A||h||^2 \leq \langle S_f(h), h\rangle \leq B||h||^2.$$

Thus, $A \leq S_f \leq B$, which implies the invertibility of S_f.

Let $f : X \to H$ be a C-Bessel mapping for H and $E \subseteq X$ be measurable. Then, it is clear that $f\chi_E : X \to H$ and $f|_E : E \to H$ are C-Bessel mappings for H. Also,

$$L^2(E) = L^2(X)|_E = \{g|_E : g \in L^2(X)\}.$$

So, we can embed $L^2(E)$ in $L^2(X)$ as a closed subspace. Since, for each $h \in H$ and each $g \in L^2(X)$

$$\int_X g(x)\langle f(x)\chi_E(x), h\rangle d\mu = \int_E g(x)\langle f(x), h\rangle d\mu,$$

and the operators $T_{f|_E}$ and $T_{f\chi_E}$ are unique, we shall identify them. Therefore, for each $E, F \subseteq X$ measurable, we have

$$T_{f|_E} + T_{f|_F} = T_{f\chi_E} + T_{f\chi_F}.$$

Hence, for disjoint E, F we have

$$T_{f|_E} + T_{f|_F} = T_{f\chi_{(E\cup F)}} = T_{f|_{(E\cup F)}}.$$

Theorem 2.5. *Let $f : X \to H$ be C-Bessel mapping for H, and let $\{E_i\}$ be a sequence of measurable subsets of X. Then :*
(i) $\lim_n ||T_{f|_{\cup_1^n E_i}}|| = ||T_{f|_{\cup_i E_i}}||$.
(ii) If the sequence $\{E_i\}$ is pairwise disjoint then

$$\sum_i T_{f|_{E_i}} = T_{f|_{\cup_i E_i}}.$$

Proof. (i) We have

$$||T^*_{f|_{\cup_i E_i}}||^2 = \sup_{h\in H_1} ||T^*_{f|_{\cup_i E_i}}(h)||^2 = \sup_{h\in H_1} \int_{\cup_i E_i} |\langle f(x), h\rangle|^2 d\mu$$

$$= \sup_{h\in H_1} \lim_n \int_{\cup_1^n E_i} |\langle f(x), h\rangle|^2 d\mu = \lim_n \sup_{h\in H_1} \int_{\cup_1^n E_i} |\langle f(x), h\rangle|^2 d\mu$$

$$= \lim_n ||T^*_{f|_{\cup_i^n E_i}}||^2 = \lim_n ||T_{f|_{\cup_1^n E_i}}||^2.$$

So

$$\lim_n ||T_{f|_{\cup_1^n E_i}}|| = ||T_{f|_{\cup_i E_i}}||.$$

(ii) Let $E = \cup E_i$. We have

$$||T_{f|_{\cup_i E_i}} - T_{f|_{\cup_1^n E_i}}||^2 = ||T_{f|_{\cup_n^\infty E_i}}||^2$$

$$= \sup_{h\in H_1} \int_{(\cup_n^\infty E_i)} |\langle f(x), h\rangle|^2 d\mu$$

$$= \sup_{h\in H_1} \int_X \sum_n^\infty \chi_{E_i}(x) |\langle f(x), h\rangle|^2 d\mu \to 0,$$

and (ii) is proved.

Theorem 2.6. *Let K be a Hilbert space, $f : X \to H$ be a C-Bessel mapping for H, and $u : H \to K$ be a bounded linear mapping. Then:*
(i) The mapping $uf : X \to K$ is a C-Bessel mapping for K, and

$$uT_f = T_{uf}.$$

(ii) Let f be a C-frame for H. Then, uf is a C-frame for K if and only if u is surjective.

Proof. (i) Since

$$\sup_{h\in H_1} \int_X |\langle h, u(f(x))\rangle|^2 d\mu \le ||u||^2 \sup_{h\in H_1} \int_X |\langle h, f(x)\rangle|^2 d\mu,$$

uf is a C-Bessel mapping for K. For each $g \in L^2(X)$, we have

$$\langle T_{uf}(g), k\rangle = \int_X g(x)\langle u(f(x)), k\rangle d\mu$$

$$= \int_X g(x)\langle f(x), u^*(k)\rangle d\mu = \langle uT_f(g), k\rangle.$$

Hence, $T_{uf} = uT_f$.
(ii) If u is surjective then by the Lemma (2.4), uT_f is surjective. So uf is a C-frame for K. Now, if uf is a C-frame for K then T_{uf} is surjective, so u is surjective.

Theorem 2.7. *Let $f : X \to H$ be a C-frame for H. Then :*
(i) $\sup_{h\in H_1} ||T_f^*(h)||^2 = ||S_f||$.
(ii) $\inf_{h\in H_1} ||T_f^*(h)||^2 = ||S_f^{-1}||^{-1}$.

Proof. (i) Straightforward.
(ii) Since

$$\inf_{h\in H_1} ||T_f^*(h)||^2 \leq S \leq \sup_{h\in H_1} ||T_f^*(h)||^2,$$

so

$$(\sup_{h\in H_1} ||T_f^*(h)||^2)^{-1} \leq S_f^{-1} \leq (\inf_{h\in H_1} ||T_f^*(h)||^2)^{-1}.$$

Since

$$T_{S_f^{-1}f} T^*_{S_f^{-1}f} = S_f^{-1} T_f T_f^* S_f^{-1} = S_f^{-1},$$

hence

$$(\sup_{h\in H_1} ||T_f^*(h)||^2)^{-1} \leq T_{S_f^{-1}f} T^*_{S_f^{-1}f} \leq (\inf_{h\in H_1} ||T_f^*(h)||^2)^{-1}.$$

Thus

$$(\sup_{h\in H_1} ||T_f^*(h)||^2)^{-1} \leq \inf_{h\in H_1} ||T^*_{S_f^{-1}f}(h)||^2, \tag{2.1}$$

$$\sup_{h\in H_1} ||T^*_{S_f^{-1}f}(h)||^2 \leq (\inf_{h\in H_1} ||T_f^*(h)||^2)^{-1}. \tag{2.2}$$

Similarly, we conclude that

$$(\sup_{||h||=1} ||T^*_{S_f^{-1}f}(h)||^2)^{-1} \leq \inf_{h\in H_1} ||T_f^*(h)||^2, \tag{2.3}$$

$$\sup_{h\in H_1} ||T_f^*(h)||^2 \leq (\inf_{h\in H_1} ||T^*_{S_f^{-1}f}(h)||^2)^{-1}. \tag{2.4}$$

Thus, by (2.2) and (2.3), we have

$$\begin{aligned}\inf_{h\in H_1} ||T_f^*(h)||^2 &= (\sup_{h\in H_1} ||T^*_{S_f^{-1}f}(h)||^2)^{-1} \\ &= ||S_{S_f^{-1}f}||^{-1} = ||S_f^{-1}||^{-1}.\end{aligned}$$

Corollary 2.8. *Let f be a C-frame for H. Then $||S_f^{-1}||^{-1}$ and $||S_f||$ are the optimal values which satisfy*

$$||S_f^{-1}||^{-1} \leq S_f \leq ||S_f||.$$

3. Duals of C-Bessel mappings

In this section we shall show more properties of C-Bessel mappings.

Definition 3.1. *Let f, g be C-Bessel mappings for H. We say that f and g are weakly equal if $T_f^* = T_g^*$, which is equivalent with*

$$\langle h, f \rangle = \langle h, g \rangle \quad a.e.$$

for all $h \in H$.

Theorem 3.2. *Let f, g be C-Bessel mappings for H. Then the following assertions are equivalent:*
(i) For each $h \in H, h = T_f(\langle h, g\rangle)$.
(ii) For each $h \in H, h = T_g(\langle h, f\rangle)$.
(iii) For each $h, k \in H, \langle h, k\rangle = \int_X \langle h, f(x)\rangle\langle g(x), k\rangle d\mu$.
(iv) For each $h \in H, ||h||^2 = \int_X \langle h, f(x)\rangle\langle g(x), h\rangle d\mu$.
(v) For each orthonormal bases $\{e_i\}_{i\in I}$ and $\{\gamma_j\}_{j\in J}$ for H

$$\langle e_i, \gamma_j\rangle = \int_X \langle e_i, f(x)\rangle\langle g(x), \gamma_j\rangle d\mu, \quad i \in I, j \in J.$$

(vi) For each orthonormal basis $\{e_i\}_{i\in I}$ for H

$$\int_X \langle e_i, f(x)\rangle\langle g(x), e_j\rangle d\mu = \langle e_i, e_j\rangle, \quad i, j \in I.$$

Proof. $(i) \to (ii)$ Let $h, k \in H$. We have

$$\langle h, k\rangle = \langle T_f(\langle h, g\rangle), k\rangle = \int_X \langle h, g(x)\rangle\langle f(x), k\rangle d\mu$$
$$= \overline{\langle T_g(\langle k, f\rangle), h\rangle} = \langle h, T_g(\langle k, f\rangle)\rangle.$$

Hence, $k = T_g(\langle k, f\rangle)$.
$(ii) \to (iii)$ It is evident by the proof of $(i) \to (ii)$.

$(iii) \to (i)$ Let $h, k \in H$. Then

$$\langle h, k\rangle = \int_X \langle h, g(x)\rangle\langle f(x), k\rangle d\mu = \langle T_f(\langle h, g\rangle), k\rangle.$$

So, $h = T_f(\langle h, g\rangle)$.

$(iv) \to (i)$ Let $F : H \to H$ be defined by

$$F(h) = T_f(\langle h, g\rangle).$$

It is clear that F is linear. Since

$$||F(h)|| = \sup_{k\in H_1} |\langle F(h), k\rangle| = \sup_{k\in H_1} |\int_X \langle h, g(x)\rangle\langle f(x), k\rangle d\mu|$$

$$\leq \sup_{k\in H_1} (\int_X |\langle k, f(x)\rangle|^2 d\mu)^{1/2} \sup_{h\in H_1} (\int_X |\langle h, g(x)\rangle|^2 d\mu)^{1/2}||h||,$$

$F \in B(H)$. For each $h \in H$, we have

$$\langle h, h\rangle = ||h||^2 = \int_X \langle g(x), h\rangle\langle h, f(x)\rangle d\mu = \langle T_f(\langle h, g\rangle), h\rangle.$$

Hence, for each $h \in H, h = T_f(\langle h, g\rangle)$.

$(iii) \to (v), (iii) \to (iv)$ and $(v) \to (vi)$ are evident.

$(v) \to (iii)$ We have

$$\int_X \langle h, f(x)\rangle\langle g(x), k\rangle d\mu = \langle\langle h, f\rangle, \langle k, g\rangle\rangle_{L^2}$$

$$= \langle\langle h, \sum_i \overline{\langle e_i, f\rangle} e_i\rangle, \langle k, \sum_j \overline{\langle \gamma_j, g\rangle}\gamma_j\rangle\rangle_{L^2}$$

$$= \sum_{i,j} \langle\langle h, \overline{\langle e_i, f\rangle} e_i\rangle, \langle k, \overline{\langle \gamma_j, g\rangle}\gamma_j\rangle\rangle_{L^2}$$

$$= \sum_{i,j} \langle h, e_i\rangle\langle \gamma_j, k\rangle\langle\langle e_i, f\rangle, \langle \gamma_j, g\rangle\rangle$$

$$= \sum_{i,j} \langle h, e_i\rangle\langle \gamma_j, k\rangle \int_X \langle e_i, f(x)\rangle\langle g(x), \gamma_j\rangle d\mu$$

$$= \sum_{i,j} \langle h, e_i\rangle\langle \gamma_j, k\rangle\langle e_i, \gamma_j\rangle = \langle h, k\rangle.$$

$(vi) \to (v)$ It is similar with the proof of $(v) \to (iii)$.

Definition 3.3. *Let f, g be C-Bessel mappings for H. We say that f, g is a dual pair, if one of the assertions of the Theorem* (3.2) *satisfies.*

Lemma 3.4. *Let f, g be a dual pair. Then f is a C-frame for H.*

Proof. For each $h \in H$, we have

$$||h||^2 \leq \int_X |\langle h, f(x)\rangle||\langle g(x), h\rangle| d\mu$$

$$\leq (\int_X |\langle h, f(x)\rangle|^2 d\mu)^{1/2} (\int_X |\langle g(x), h\rangle|^2 d\mu)^{1/2}$$

$$\leq (\sup_{h\in H_1} ||T_g^*(h)||^2)^{1/2} ||h|| (\int_X |\langle h, f(x)\rangle|^2 d\mu)^{1/2}.$$

Since, f, g is a dual pair, $g \neq 0$ weakly. So

$$\sup_{h\in H_1} ||T_g^*(h)|| > 0.$$

Thus

$$(\sup_{h\in H_1} ||T_g^*(h)||^2)^{-1} ||h||^2 \leq \int_X |\langle h, f(x)\rangle|^2 d\mu.$$

Hence, f is a C-frame for H.

Theorem 3.5. *Let f be a C-frame for H. Let $h \in H$. Then :*
(i) We have the following retrieval formulas

$$h = T_{S_f^{-1} f}(\langle h, f\rangle),$$

$$h = T_f(\langle S_f^{-1}(h), f\rangle).$$

(ii) In the retrieval formula $h = T_f(\langle S_f^{-1}(h), f\rangle$, $\langle h, S_f^{-1} f\rangle$ has least norm among all of the retrieval formulas.
(iii) For each $h \in H, h = T_f\langle h, g\rangle$ if and only if there exists a C-Bessel mapping l for H such that $g = S_f^{-1} f + l$ where for each $k \in H, \langle k, l\rangle \in ker T_f$.
(iv) f has just one dual if and only if

$$\mathcal{R}T_f^* = L^2(X).$$

Proof. (i) Let $k \in H$. Since, S_f is an invertible operator, we have

$$h = S_f^{-1} S_f(h) = S_f^{-1} T_f T_f^*(h) = T_{S_f^{-1} f}(\langle h, f\rangle).$$

Also, we have

$$h = S_f S_f^{-1}(h) = T_f \langle S_f^{-1}(h), f\rangle.$$

(ii) Let $\phi \in L^2(X)$ and $h = T_f(\phi)$. Thus, for each $g \in H$, we have

$$\langle h, g\rangle = \langle T_f(\langle S_f^{-1}(h), f\rangle), g\rangle = \int_X \langle S_f^{-1}(h), f(x)\rangle\langle f(x), g\rangle d\mu.$$

By the way, we have

$$\langle h, g\rangle = \langle T_f(\phi), g\rangle = \int_X \phi(x)\langle f(x), g\rangle d\mu.$$

Therefore

$$\langle T_f(\langle S_f^{-1}(h), f\rangle - \phi), g\rangle$$
$$= \int_X (\langle S_f^{-1}(h), f(x)\rangle - \phi(x))\langle f(x), g\rangle d\mu = 0.$$

So

$$T_f(\langle S_f^{-1}(h), f\rangle - \phi) = 0.$$

Hence, $\langle S_f^{-1}(h), f\rangle - \phi \in ker(T_f)$. Since, f is a C-Bessel mapping for H

$$\langle S_f^{-1}(h), f\rangle \in \mathcal{R}T_f^*.$$

But, $L^2(X) = (kerT_f) \oplus (\mathcal{R}T_f^*)$. So

$$||\phi||^2 = ||\langle S_f^{-1}(h), f\rangle - \phi||^2 + ||\langle S_f^{-1}(h), f\rangle||^2,$$

and (ii) is proved.
(iii) Let g be a C-Bessel mapping for H, and let for each $h \in H, h = T_f\langle h, g\rangle$. Let $g - S_f^{-1}f = l$. By the Theorem (3.2), for each $h, k \in H$, we have

$$\langle\langle k, l\rangle, \langle h, f\rangle\rangle = \langle\langle k, g\rangle, \langle h, f\rangle\rangle - \langle\langle k, S^{-1}f\rangle, \langle h, f\rangle\rangle$$
$$= \langle k, h\rangle - \langle k, h\rangle = 0.$$

Hence, for each $k \in H, \langle k, l\rangle \in (\mathcal{R}T_f^*)^{\perp} = kerT_f$.
Now, let $g = S_f^{-1}f + l$ where for each $k \in H, \langle k, l\rangle \in \ker T_f$. We have

$$\int_X \langle h, f(x)\rangle\langle S_f^{-1}f(x) + l(x), k\rangle d\mu = \langle\langle h, f\rangle, \langle k, S_f^{-1}f + l\rangle\rangle_{L^2}$$
$$= \langle\langle h, f\rangle, \langle k, S_f^{-1}f\rangle\rangle_{L^2} + \langle\langle h, f\rangle, \langle k, l\rangle\rangle_{L^2}$$
$$= \langle\langle h, f\rangle, \langle k, S_f^{-1}f\rangle\rangle_{L^2} = \int_X \langle h, f\rangle\langle S^{-1}f, k\rangle d\mu = \langle h, k\rangle.$$

Thus, by the Theorem(3.2), for each $h \in H, h = T_f\langle h, g\rangle$.
(iv) Let $\mathcal{R}T_f^* \neq L^2(X)$. Let $l \in (\mathcal{R}T_f^*)^\perp$ with $||l|| = 1$. Let $k : X \to L^2(X)$ be defined by $k(x) = l(x)l$. For each $t \in L^2(X)$, the mapping

$$X \to \mathbb{C}, \quad x \mapsto \langle t, k(x)\rangle = \langle t, l\rangle l(x),$$

is measurable and

$$\int_X |\langle t, k(x)\rangle|^2 d\mu = \int_X |\langle t, l\rangle|^2 |l(x)|^2 d\mu$$
$$= |\langle t, l\rangle|^2 \leq ||t||^2.$$

Thus k is a C-Bessel mapping for $L^2(X)$. Let $v : L^2(X) \to H$ be such that $v(l) \neq 0$. Then vk is a C-Bessel mapping for H, and so $S^{-1}f + vk$ is a C-Bessel mapping for H. Let $h \in H$. Since

$$\int_X \langle h, S_f^{-1}f(x) + vk(x)\rangle\langle f(x), h\rangle d\mu$$
$$= \int_X \langle h, S_f^{-1}f(x)\rangle\langle f(x), h\rangle d\mu + \int_X \langle h, vk(x)\rangle\langle f(x), h\rangle d\mu$$
$$= ||h||^2 + \langle v^*(h), l\rangle \int_X \overline{l(x)}\langle f(x), h\rangle d\mu = ||h||^2,$$

$S_f^{-1}f + vk$ is a dual of f. Since,

$$\langle v(l), vk(x)\rangle = \langle v(l), l(x)v(l)\rangle = \overline{l(x)}\langle v(l), v(l)\rangle,$$

$S_f^{-1}f + vk$ is not weakly equal to $S_f^{-1}f$. Conversely, let $L^2(X) = \mathcal{R}T_f^*$. Now, let $g = S_f^{-1}f + l$ where for each $k \in H$

$$\langle k, l\rangle \in kerT_f = (\mathcal{R}T_f^*)^\perp = \{0\}.$$

Therefore, $l = 0$ weakly, so f has just one dual.

Theorem 3.6. *Let f be a C-Bessel mapping for H and $f \neq 0$ weakly. We have:*
(i) *If k is a C-Bessel mapping for H then the mapping $U : X \times X \to \mathbb{C}$ defined by*

$$U(x, y) = \langle f(x), k\rangle(y) = \langle f(x), k(y)\rangle,$$

defines a bounded operator on $L^2(X)$.
(ii) *Let $U : X \times X \to \mathbb{C}$ defines a bounded operator $W : L^2(X) \to L^2(X)$ as (i). Let g of X to H be defined by*

$$g(x) = T_f(U(x, .)).$$

Then g is defined for almost all $x \in X$ and g is a C-Bessel mapping for H. Let f be a C-frame for H then g is a C-frame for H if and only if there exists $c > 0$ such that

$$\inf_{h\in H_1} ||T_f^*(h)|| \leq c \inf_{h\in H_1} ||T_g^*(h)||.$$

Proof. (i) Let $l \in L^2(X)$ and $x \in X$. We define

$$W_l(x) = \int_X U(x,y)l(y)d\mu_y.$$

Since, f, k are C-Bessel mappings for H and $\overline{W}_l(x) = \langle T_k(\bar{l}), f(x)\rangle$, W_l is measurable. Also, we have

$$\int_X |W_l(x)|^2 d\mu_x = \int_X |\langle T_k(\bar{l}), f(x)\rangle|^2 d\mu_x = ||T_f^*(T_k(\bar{l})||^2$$
$$\leq ||T_f||^2 ||T_k(\bar{l})||^2 \leq ||T_f||^2 ||T_k||^2 ||l||^2.$$

Thus, $W : L^2(X) \to L^2(X)$ defined by $W(l) = W_l$ is a bounded operator.
(ii) Since

$$||W_l|| = \int_X |W_l(x)|^2 d\mu_x$$
$$= \int_X |\int_X U(x,y)l(y)d\mu_y|^2 d\mu_x \leq ||W|| ||l||,$$

for almost all $x \in X$, $U(x,.)l \in L^1(X)$. So, for almost all $x \in X$, $U(x,.) \in L^2(X)$. Hence, g is defined for almost all $x \in X$. Since

$$\overline{\langle h, g(x)\rangle} = \int_X U(x,y)\langle f(y), h\rangle d\mu_y = W_{\overline{\langle h,f\rangle}}(x),$$

g is weakly measurable. But

$$\int_X |\langle h, g(x)\rangle|^2 d\mu_x = \int_X |W_{\overline{\langle h,f\rangle}}(x)|^2 d\mu_x \leq ||W|| ||\langle h, f\rangle||.$$

So, g is a C-Bessel mapping for H. If g is a C-frame for H then

$$(\inf_{h\in H_1} ||T_g^*(h)||^2 / \sup_{h\in H_1} ||T_f^*(h)||^2) \inf_{h\in H_1} ||T_f^*(h)||^2 ||h||^4$$
$$\leq (\inf_{h\in H_1} ||T_g^*(h)||^2 / \sup_{h\in H_1} ||T_f^*(h)||^2) ||T_f^*||^2 ||h||^4$$
$$= \inf_{h\in H_1} ||T_g^*(h)||^2 ||h||^4.$$

Thus

$$\inf_{h\in H_1} ||T_f^*(h)|| \leq c \inf_{h\in H_1} ||T_g^*(h)||,$$

where
$$c = (\inf_{h\in H_1}(||T_g^*(h)||^2/\sup_{h\in H_1}||T_f^*(h)||^2)^{-1/2} > 0.$$
The converse is clear.

Theorem 3.7. *Let f be a C-frame for H. Then:*
(i) Let $l \in L^2(X)$. If $h = T_f(l)$ then
$$||l||^2 = \int_X |\langle h, S_f^{-1}f(x)\rangle|^2 d\mu_x + \int_X |l(x) - \langle h, S_f^{-1}f(x)\rangle|^2 d\mu_x.$$
(ii) For each $h \in H, T_f^\dagger(h) = \langle h, S_f^{-1}f\rangle$.
(iii) $||T_f^\dagger||^{-2} = \inf_{h\in H_1}||T_f^*(h)||^2$.

Proof. (i) By the Theorem (3.5), $T_f(l - \langle h, S_f^{-1}f\rangle) = 0$. So
$$l - \langle h, S_f^{-1}f\rangle \in \ker T_f = (\mathcal{R}T_f^*)^\perp.$$
Since, $\langle h, S_f^{-1}f\rangle \in \mathcal{R}T_f^*$,
$$||l||^2 = ||l - \langle h, S_f^{-1}f\rangle||_2^2 + ||\langle h, S_f^{-1}f\rangle||_2^2.$$

(ii) Since, $T_f^\dagger(h)$ is the unique solution of minimal norm of
$$T_f(l) = h,$$
so
$$\int_X |\langle l(x) - \langle h, S_f^{-1}f(x)\rangle|^2 d\mu_x = 0.$$
Hence, $l = \langle h, S_f^{-1}f\rangle = T_f^\dagger(h)$.
(iii) Since, f is a C-frame for H, by the Lemma (2.4), $S_f^{-1}f$ is a C-frame for H. Therefore
$$||T_f^\dagger||^2 = \sup_{h\in H_1}\int_X |\langle h, S_f^{-1}f(x)\rangle|^2 d\mu_x = ||T_{S^{-1}f}T_{S^{-1}f}^*||$$
$$= ||S_f^{-1}|| = (\inf_{h\in H_1}||T_f^*(h)||^2)^{-1}.$$

Theorem 3.8. *Let f be a C-frame for H and $g \in L^2(X)$. Then, $h = T_{S^{-1}f}(g)$ is the unique vector in H which minimizes the mapping*
$$H \to \mathbb{C}, \quad h \mapsto \int_X |g - \langle h, f\rangle|^2 d\mu.$$

Proof. Since, $\mathcal{R}T_f^*$ is closed and

$$\int_X |g - \langle h, f\rangle|^2 d\mu = ||g - \langle h, f\rangle||_2^2,$$

it is enough to prove that the mapping

$$L^2(X) \to L^2(X), \quad g \mapsto \langle T_{S^{-1}f}(g), f\rangle$$

is the orthonormal projection of $L^2(X)$ onto $\mathcal{R}T_f^*$.
Let $g \in \mathcal{R}T_f^{*\perp}$. Then

$$\langle T_{S_f^{-1}f}(g), f\rangle = \langle S_f^{-1}T_f(g), f\rangle = \langle T_f(g), S_f^{-1}f\rangle = 0,$$

because, for each $x \in X$,

$$\begin{aligned}\langle T_f(g), S_f^{-1}f\rangle(x) &= \int_X g(y)\langle f(y), S_f^{-1}f(x)\rangle d\mu_y \\ &= \langle g, \langle S_f^{-1}f(x), f\rangle\rangle_{L^2} = 0.\end{aligned}$$

Now, let $g \in \mathcal{R}T_f^*$. So, there exists $h \in H$ with $g = \langle h, f\rangle$. We have,

$$\langle T_{S_f^{-1}f}(g), f\rangle = \langle S_f^{-1}S(h), f\rangle = \langle h, f\rangle = g,$$

and the theorem is proved.

Definition 3.9. *Let f, g be C-Bessel mappings for H. We define $\langle f, g\rangle_{\mathcal{B}} : X \to L^2(X)$ by*

$$\langle f, g\rangle_{\mathcal{B}}(x) = \langle f(x), g\rangle.$$

Theorem 3.10. *Let f, g be C-Bessel mappings for H. Then :*
(i) $T_g^* f = \langle f, g\rangle_{\mathcal{B}}$.
(ii) $\langle f, g\rangle_{\mathcal{B}}$ is a C-Bessel mapping for $L^2(X)$.
(iii) Let f be a C-frame for H and $K = \mathcal{R}T_g^*$ be closed. Then $\langle f, g\rangle_{\mathcal{B}}$ is a C-frame for $L^2(X)$ and there exists a surjective bounded operator $u : L^2(X) \to H$ such that $g = u\langle g, f\rangle_{\mathcal{B}}$.

Proof. (i) Let $l \in L^2(X)$. For each $x \in X$, we have

$$\begin{aligned}\langle l, T_g^* f(x)\rangle = \langle T_g(l), f(x)\rangle &= \int_X l(y)\langle g(y), f(x)\rangle d\mu_y \\ = \int_X l(y)\overline{\langle f(x), g(y)\rangle} d\mu_y &= \langle l, \langle f(x), g\rangle\rangle_{L^2} \\ &= \langle l, \langle f, g\rangle_{\mathcal{B}}(x)\rangle_{L^2}.\end{aligned}$$

Thus $T_g^* f = \langle f, g\rangle_{\mathcal{B}}$.
(ii) Let $l \in L^2(X)$. Since, the mapping

$$X \to \mathbb{C}, \quad x \mapsto \langle l, \langle f, g\rangle_{\mathcal{B}}(x)\rangle = \langle l, T_g^* f(x)\rangle = \langle T_g(l), f(x)\rangle$$

is measurable, $\langle f, g\rangle_{\mathcal{B}}$ is weakly measurable. Since

$$\int_X |\langle l, \langle f, g\rangle_{\mathcal{B}}(x)\rangle|^2 d\mu_x = \int_X |\langle l, T_g^*(f(x))\rangle|^2 d\mu_x$$

$$= \int_X |\langle T_g(l), f(x)\rangle|^2 d\mu_x \leq \sup_{h\in H_1} ||T_f^*(h)||^2 ||T_g(l)||^2$$

$$\leq ||T_f||^2 ||T_g||^2 ||l||^2,$$

$\langle f, g\rangle_{\mathcal{B}}$ is a C-Bessel mapping for $L^2(X)$.
(iii) For each $l \in \mathcal{R}T_g^*$, we have

$$||l|| = ||T_g^*(T_g^*)^\dagger(l)|| = ||((T_g^*)^\dagger)^*(T_g(l))||$$

$$\leq ||T_g^\dagger||.||T_g(l)||.$$

Hence

$$||T_g^\dagger||^{-1}||l|| \leq ||T_g(l)||.$$

Thus

$$\inf_{h\in H_1} ||T_f^*(h)||^2 ||T_g^\dagger||^{-2}||l||^2 \leq \inf_{h\in H_1} ||T_f^*(h)||^2 ||T_g(l)||^2$$

$$\leq \int_X |\langle T_g(l), f(x)\rangle|^2 d\mu_x = \int_X |\langle l, \langle f, g\rangle_{\mathcal{B}}(x)\rangle|^2 d\mu_x.$$

Hence

$$\inf_{l\in K_1} ||T^*_{\langle f,g\rangle_{\mathcal{B}}}(l)|| > 0,$$

and $\langle h, g\rangle_{\mathcal{B}}$ is a frame for $L^2(X)$. We have the following retrieval formula

$$g = S_f^{-1} T_f T_f^* g = S_f^{-1} T_f(\langle g, f\rangle_{\mathcal{B}}).$$

So, $g = u\langle g, f\rangle_{\mathcal{B}}$, where $u = S_f^{-1}T_f$ is a bounded surjective operator of $L^2(X)$ to H.

Theorem 3.11. *Let $f, g \in L^2(X, H)$ and let $t = T_f T_g^*$ then:*
(i) The mapping t is a trace class operator.
(ii) If μ is a σ-finite measure then $tr(t) = \langle f, \overline{g}\rangle_{L^2}$.

Proof. (i) Let $t = u|t|$ be the polar decomposition of t. So, $|t| = u^*t$. Thus $|t| = T_{u^*f}T_g^*$. Let $\{e_j\}_{j\in J}$ be an orthonormal basis for H. We have

$$tr(|t|) = \sum_{j\in J}\langle |t|(e_j), e_j\rangle \leq \sum_{j\in J}\int_X |\langle e_j, g(x)\rangle||\langle u^*f(x), e_j\rangle| d\mu$$

$$\leq \int_X (\sum_j |\langle e_j, g(x)\rangle|^2)^{1/2}(\sum_j |\langle u^*f(x), e_j\rangle|^2)^{1/2} d\mu$$

$$= \int_X ||u^*f(x)||||g(x)||d\mu \leq \int_X ||f(x)||||g(x)||d\mu$$

$$\leq ||f||_2||g||_2.$$

Thus, t is a trace class operator.
(ii) Let $\{e_j\}_{j\in J}$ be an orthonormal basis for H. We have

$$tr(t) = \sum_j \langle T_f T_g^*(e_j), e_j\rangle = \sum_j \int_X \langle e_j, g(x)\rangle\langle f(x), e_j\rangle d\mu.$$

Since

$$\int_X \sum_j |\langle e_j, g(x)\rangle\langle f(x), e_j\rangle| d\mu$$

$$\leq \int_X (\sum_\alpha |\langle e_j, g(x)\rangle|^2)^{1/2}(\sum_j |\langle f(x), e_j\rangle|^2)^{1/2} d\mu$$

$$\leq ||g||_2||f||_2,$$

so

$$tr(t) = \int_X \sum_j \langle e_j, g(x)\rangle\langle f(x), e_j\rangle d\mu$$

$$= \int_X \langle f(x), g(x)\rangle d\mu = \langle f, \overline{g}\rangle_{L^2}.$$

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